Inventory Theory. S5

The \((s, Q)\) Inventory Policy

We consider now inventory systems similar to the deterministic models, however, we allow demand to be stochastic. There are a number of ways one might operate an inventory system with random demand. In this section, we consider the \((s, Q)\) inventory policy, alternatively called the reorder point, order quantity system. Fig. 9 shows the inventory pattern determined by the \((s, Q)\) inventory policy. The model assumes that the inventory level is observed at all times. This is called continuous review. When the level declines to some specified reorder point, \(s\), an order is placed for a lot size, \(Q\). The order arrives to replenish the inventory after a lead time, \(L\).

![Figure 9. Inventory Operated with the Reorder Point-Lot Size Policy](image)

**Model**

The values of \(s\) and \(Q\) are the two decisions required to implement the policy. The lead time is assumed known and constant. The only uncertainty is associated with demand. In the figure we show the decrease in inventory level between replenishments as a straight line, but in reality the inventory decreases in a stepwise and uneven fashion due to the discrete and random nature of the demand process.

If we assume that \(L\) is relatively small compared to the expected time required to exhaust the quantity \(Q\), it is likely that only one order is outstanding at any one time. This is the case illustrated in the figure. We call the period between sequential order arrivals an order cycle. The cycle begins with the receipt of the lot, it progresses as demand depletes the inventory to the level \(s\), and then it continues for the time \(L\) when the next lot is received. As we see in the figure, the inventory level increases instantaneously by the amount \(Q\) with the receipt of an order.
In the following analysis we are most concerned with the possibility of shortage during an order cycle, that is the event of the inventory level falling below zero. This is also called the stockout event. We assume shortages are backordered and are satisfied when the next replenishment arrives. To determine probabilities of shortages, one need only be concerned about the random variable that is the demand during the lead time interval. This is the random variable $X$ with p.d.f., $f(x)$, and c.d.f. $F(x)$. The mean and standard deviation of the distribution are $\mu$ and $\sigma$ respectively. The random demand during the lead time gives rise to the possibility that the inventory level will be depleted before the replenishment arrives. With the average rate of demand equal to $a$, the mean demand during the lead time is

$$\mu = aL$$

A shortage will occur if the demand during the period $L$ is greater than $s$. This probability, defined as $P_s$, is

$$P_s = P\{x > s\} = \int_{s}^{\infty} f(x)dx = 1 - F(s).$$

The service level is the probability that the inventory will not be depleted during one order cycle, or

$$Service\ level = 1 - P_s = F(s).$$

In practical instances the reorder point is significantly greater than the mean demand during the lead time so that $P_s$ is quite small. The safety stock, $SS$

$$SS = s - \mu.$$ 

This is the inventory maintained to protect the system against the variability of demand. It is the expected inventory level at the end of an order cycle (just before a replenishment arrives). This is seen in Fig. 10, where we show the $(s, Q)$ policy for deterministic demand. This figure will also be useful for the cost analysis of the system.
The General Solution for the \((s, Q)\) Policy

We develop here a general cost model for the \((s, Q)\) policy. The model and its optimum solution depends on the assumption we make regarding the cost effects of shortage. The model is approximate in that we do not explicitly model all the effects of randomness. The principal assumption is that stockouts are rare, a practical assumption in many instances. In the model we use the same notation as for the deterministic models of Section 23.2. Since demand is a random variable, we use \(a\) as the time averaged demand rate per unit time.

When we assume that the event of a stockout is rare and inventory declines in a continuous manner between replenishments, the average inventory is approximately

\[
\text{Average Inventory Level} = \frac{Q}{2} + s - \mu.
\]

Since the per unit holding cost is \(h\), the holding cost per unit time is

\[
\text{Expected Holding Cost per unit time} = h\left(\frac{Q}{2} + s - \mu\right).
\]

With the backorder assumption, the time between orders is random with a mean value of \(Q/a\). The cost for replenishment is \(K\), so the expected replenishment cost per unit time is

\[
\text{Expected Replenishment Cost per unit time} = \frac{K a}{Q}.
\]

With the \((s, Q)\) policy and the assumption that \(L\) is relatively smaller than the time between orders, \(Q/a\), the shortage cost per cycle depends only on the reorder point. We call this \(C_s\), and we observe that it is a function of the reorder point \(s\).
We investigate several alternatives for the definition of this shortage cost. Dividing this cost by the length of a cycle we obtain

\[ \text{Expected Shortage Cost per unit time} = \frac{a}{Q} C_s. \]

Combining these terms we have the general model for the expected cost of the \((s, Q)\) policy.

\[
\begin{align*}
EC(s, Q) &= h\left(\frac{Q}{2} + s - \mu\right) \quad \text{Inventory Cost} \\
&\quad + \frac{K a}{Q} \quad \text{Replenishment Cost} \\
&\quad + \frac{a}{Q} C_s \quad \text{Shortage Cost} \\
&= h\left(\frac{Q}{2} + s - \mu\right) + \frac{K a}{Q} + \frac{a}{Q} C_s \\
&= h\left(\frac{Q}{2} + s - \mu\right) \quad (37)
\end{align*}
\]

There are two variables in this cost function, \(Q\) and \(s\). To find the optimum policy that minimizes cost, we take the partial derivatives of the expected cost, Eq. 37, with respect to each variable and set them equal to zero. First, the partial derivative with respect to \(Q\) is

\[
\frac{\partial EC}{\partial Q} = h\left(\frac{1}{2} - \frac{a(K + C_s)}{Q^2}\right) = 0.
\]

or \(Q^* = \sqrt{\frac{2a(K + C_s)}{h}} \) \hspace{1cm} (38)

We have a general expression for the optimum lot size that depends on the cost due to shortages.

Taking the partial derivative with respect to the variable \(s\),

\[
\frac{\partial EC}{\partial s} = h + \frac{a}{Q}\left(\frac{\partial C_s}{\partial s}\right) = 0,
\]

or \(\frac{\partial C_s}{\partial s} = -\frac{hQ}{a} \) \hspace{1cm} (39)

The solution for the optimum reorder point depends on the functional form of the cost of shortage. We consider four different cases in the remainder of this section\(^1\).

\(^1\)In this article we follow the development in Silver and Peterson, Chapter 7.
The Case of a Fixed Cost per Stockout

In this case there is a cost $\pi_1$ expended whenever there is the event of a stockout. This cost is independent of the number of items short, just on the fact that a stockout has occurred. The expected cost per cycle is

$$C_s = \pi_1 P\{x > s\} = \pi_1 \int_s^\infty f(x) \, dx.$$  

(40)

Now the partial derivative of Eq. 40 with respect to $s$ is

$$\frac{\partial C_s}{\partial s} = -\pi_1 f(s).$$

Combining Eq. 39 with Eq. 40, we have for the optimum value of $s$

$$\frac{\partial C_s}{\partial s} = -\pi_1 f(s^*) = -\frac{hQ}{a},$$

or $f(s^*) = \frac{hQ}{\pi_1 a}$,  

(41)

and

$$C_s = \pi_1 [1 - F(s^*)].$$

(42)

Eq. 41 is a condition on the value of the p.d.f. at the optimum reorder point. If no values of the p.d.f. satisfy this equality, select some minimum safety level as prescribed by management. The p.d.f. may satisfy this condition at two different values. It can be shown that the cost function is minimized when $f(x)$ is decreasing, so for a unimodal p.d.f., select the greater of the two solutions.

Eq. 41 specifying the optimum $s^*$ together with the Eq. 38 for $Q^*$ define the optimum control parameters. If one of the parameters are given at a perhaps not optimum value, these equations yield the optimum for the other parameter. If both parameters are flexible, a successive approximation method, as illustrated in Example 13, is used to find values of $Q$ and $s$ that solve the problem.

Example 8: Optimum reorder point given the order quantity ($\pi_1$ Given)

The monthly demand for a product has a Normal distribution with a mean of 100 and a standard deviation of 20. We adopt a continuous review policy in which the order quantity is the average demand for one month. The interest rate used for time value of money calculations is 12% per year. The purchase cost of the product is $1000. When it is necessary to backorder, the cost of paperwork is estimated to be $200, independent of the number backordered. Holding cost is
estimated using the interest cost of the money invested in a unit of inventory. The lead time for this situation is 1 week. The fixed order cost is $800. Find the optimum inventory policy.

We must first adopt a time dimension for those data items related to time. Here we use 1 month. For this selection,

\[ a = 100 \text{ units/month}. \]
\[ h = 1000(0.01) = $10/\text{unit-month}, \text{ the unit cost multiplied by the interest rate. The interest rate is } 12\%/12 = 1\% \text{ per month}. \]
\[ \pi_1 = $1000, \text{ the backorder cost, which is independent in time and number.} \]
\[ K = $800, \text{ the order cost.} \]

We must also describe the distribution of demand during the lead time. For convenience we assume that 1 month has 4 weeks and that the demands in the weeks are independent and identically distributed normal variates. With these assumptions the weekly demand has

\[ \mu = 100/4 = 25, \text{ and } \sigma^2 = 20^2/4 = 100 \text{ or } \sigma = 10. \]

The problem specifies the value of \( Q \) as 1 month's demand; thus \( Q = 100. \) Using this value in Eq. 41, we find the associated optimum reorder point.

\[
\text{or } f(s^*) = \frac{hQ}{\pi_1 a} = \frac{(10)(100)}{(1000)(100)} = 0.01. \\
\]

The p.d.f. of the Standard Normal distribution is related to a general Normal distribution as

\[ f(s) = (1/\sigma)\phi(k) \text{ or } \phi(k) = \sigma f(s) \]

Then in terms of the Standard Normal we have

\[ \phi(k^*) = \sigma \frac{hQ}{\pi_1 a} = (10)(0.01) = 0.1. \]

We look this up in the Standard Normal table provided at the end of this chapter to discover \( k^* = \pm 1.66. \) Taking the larger of the two possibilities we find

\[ s^* = \mu + (1.66)\sigma = 25 + 1.66 (10) = 41.6 \]

or 42 (conservatively rounded up).

This is the optimum reorder point for the given value of \( Q. \)

**The Case of a Charge per Unit Short**

In some cases we may also be interested in the expected number of items backordered during an order cycle, \( E_s. \) This depends on the demand during the lead time.
Items backordered = \[ \begin{cases} 0 \text{ if } x \leq s \\ x - s \text{ if } x > s \end{cases} \]

Here \( E_s \) is the expected shortage and is

\[
E_s = \int_{x}^{\infty} (x - s) f(x) \, dx.
\]

For this situation, we assume a cost \( \pi_2 \) is expended for every unit short in a stockout event. The expected cost per cycle is

\[
C_s = \pi_2 E_s.
\]

Now the partial derivative with respect to \( s \) is

\[
\frac{\partial C_s}{\partial s} = -\pi_2 \left( \int_{s}^{\infty} f(x) \, dx \right) = -\pi_2 (1 - F(s)).
\]

From Eq. 41, the optimum value of \( s \) must satisfy

\[
\frac{\partial C_s}{\partial s} = -\pi_2 (1 - F(s^*)) = -\frac{hQ}{a}
\]

or

\[
F(s^*) = 1 - \frac{hQ}{\pi_2 a}.
\]

(43)

In this case we have a condition on the c.d.f. at the optimum reorder point. If the expression on the right is less than zero, use some minimum reorder point specified by management.

For a given value of \( s \), the optimum order quantity is determined from Eq. 38 by substituting the value of \( C_s \).

\[
C_s = \pi_2 E_s = \pi_2 \int_{s}^{\infty} (x - s^*) f(x) \, dx.
\]

(44)

This integral is difficult to compute except for simple distributions. It is evaluated with tables for the Normal random variable using Eq. 25.

Managers may find it difficult to specify the shortage cost \( \pi_2 \). It is easier to specify that the inventory meet some service level. One might require that the inventory meet demands from stock in 99% of the inventory cycles. The service level is actually the value of \( F(s) \). Given values of \( h, Q \) and \( a \), one can compute with Eq. 43 the implied shortage cost for the given service level.
**Example 9: Optimum reorder point given the order quantity ($\pi_2$ Given)**

We consider again Example 8, but change the cost structure for backorders. Now we assume that we must treat each backordered customer separately. The cost of paperwork and good will is estimated to be $200 per unit backordered. This is $\pi_2$. The optimum policy is governed by Eq. 43.

$$
\text{or } F(s^*) = 1 - \frac{hQ}{\pi_2 a} = 1 - \frac{(10)(100)}{(200)(100)} = 0.95.
$$

We know that the probabilities for a Normal distribution is related to the standard Normal by

$$
F(s) = \Phi\left(\frac{s - \mu}{\sigma}\right).
$$

$$
\phi(k^*) = 0.95.
$$

From a normal table we find that this is associated with a standard normal deviant of $z = 1.64$. The reorder point is then

$$
s^* = \mu + (1.64)\sigma = 25 + 1.64(10) = 41.4
$$

or 42 (conservatively rounded up).

This is the optimum for the given value of $Q$.

**The Case of a Charge per Unit Short per Unit Time**

When the backorder cost depends not only on the number of backorders but the time a backorder must wait for delivery, we would like to compute the expected unit-time of backorders for an inventory cycle. When the number of backorders is $x - s$ and the average demand rate is $a$, the average time a customer must wait for delivery is

$$
\frac{x - s}{2a}.
$$

The resulting unit-time measure for backorders is

$$
\frac{(x - s)^2}{2a}.
$$

Integrating we find the expected value, $T_s$, where

$$
T_s = \frac{1}{2a} \int_s^\infty (x - s)^2 f(x) \, dx.
$$

(45)
We consider here the case when a cost $π_3$ is expended for every unit short per unit of time. The expected cost per cycle is

$$C_s = π_3 T_s. \quad (46)$$

Now the partial derivative of $C_s$ with respect to $s$ is

$$\frac{∂C_s}{∂s} = -\frac{π_3}{a} \left( \int_s^∞ (x - s)f(x) \, dx \right) = -\frac{π_3 E_s}{a}. \quad (46)$$

From Eq. 41, the optimum value of $s$ must satisfy

$$\frac{∂C_s}{∂s} = -\frac{π_3 E_s}{a} = -\frac{hQ}{a}$$

or

$$E_s(s^*) = \frac{hQ}{π_3}. \quad (47)$$

We have added the $(s^*)$ to the expected shortage to indicate its value is a function of the reorder point$^2$.

**Example 10: Optimum reorder point given the order quantity ($π_3$ Given)**

We consider again Example 8, but now we assume that $1000 is expended per unit backorder per month. This is $π_3$. The optimum policy is governed by Eq. 47.

$$E_s(s^*) = \frac{hQ}{π_3} = \frac{(10)(100)}{1000} = 1.$$  

When the demand is governed by the Normal distribution, the expected shortage at the optimum is

$$E_s(s^*) = σG(k^*) = 1$$

where $k^* = \frac{s^* - μ}{σ}$

or $G(k^*) = 0.1$

From the table at the end of the chapter

$$k^* = 0.9.$$  

The reorder point is then

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$^2$Silver and Peterson report the more accurate result $E_s(s^*) = \frac{Qh}{h + π_3}$. This can be derived using a more accurate representation of the average inventory. The two results are approximately the same when $π_3 \gg h$, as assummed here.
\[ s^* = \mu + (0.9)\sigma = 25 + 9 = 34 \]

This is the optimum for the given value of \( Q \).

**The Lost Sales Case**

In this case sales are not backordered. A customer that arrives with no inventory on hand leaves without satisfaction, and the sale is lost. When stock is exhausted during the lead time, the inventory level rises to the level \( Q \) when it is finally replenished. The effect of this situation is to raise the average inventory level by the expected number of shortages in a cycle, \( E_s \). We also experience a shortage cost based on the number of shortages in a stockout event. We use \( \pi_L \) to indicate the cost for each lost sale. For the case of lost sales the approximate expected cost is

\[
EC(Q, s) = h\left(\frac{Q}{2} + s - \mu + E_s\right) + \frac{aK}{Q} + \frac{a\pi_L}{Q} E_s
\]

(48)

Here we are neglecting the fact that with lost sales, not all the demand is met. The number of orders per unit time is slightly less than \( a/Q \). Taking partial derivatives with respect to \( Q \) and \( s \) we find the optimum lot size is

\[
Q^* = \sqrt{\frac{2a(K + \pi_L E_s)}{h}}
\]

(49)

\[
\frac{\partial EC}{\partial s} = h(1 + \frac{\partial E_s}{\partial s}) + \frac{a\pi_L}{Q} \left(\frac{\partial E_s}{\partial s}\right) = 0,
\]

or

\[
\frac{\partial E_s}{\partial s} = -\frac{hQ}{hQ + \pi_L a}
\]

or

\[
(1 - F(s^*)) = \frac{hQ}{hQ + \pi_L a}
\]

\[
F(s^*) = 1 - \frac{hQ}{hQ + \pi_L a} = \frac{\pi_L a}{hQ + \pi_L a}
\]

(50)
**Example 11: Optimum reorder point given the order quantity (\(\pi_L\) Given)**

We consider Example 8 again, but now we assume that the sale is lost given a stockout. We charge $2000 for every lost sale. This is \(\pi_L\). The optimum policy is governed by Eq. 50.

\[
F(s^*) = \frac{\pi_L a}{hQ + \pi_L a} = \frac{(2000)(100)}{(10)(100) + (2000)(100)} = 0.995
\]

From the table at the end of the chapter

\[k^* = 2.58.\]

The reorder point is then

\[s^* = \mu + (2.58)\sigma = 25 + 22.6 = 47.6\]

This is the optimum for the given value of \(Q\).

**Summary**

We have found solutions for several assumptions regarding the costs due to shortages. These are summarized below for easy use. The optimum reorder point requires one to find the value \(s^*\) that corresponds to \(f(s^*)\), \(F(s^*)\) or \(E_s(s^*)\) equaling some simple function of the problem parameters.

The optimum order quantity for each case depends on the shortage cost, \(C_s\), and is given by

\[Q^* = \sqrt{\frac{2a(K + C_s)}{h}}\]

This equation is used directly when a value of \(s\) is specified. It is used iteratively when the optimum for both \(s\) and \(Q\) is required.

**Table 1. The \((s, Q)\) Policy for Continuous Distributions**

<table>
<thead>
<tr>
<th>Situation</th>
<th>(C_s)</th>
<th>Optimum reorder point</th>
<th>Normal Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Cost per Stockout ((\pi_1))</td>
<td>(\pi_1[1 - F(s)])</td>
<td>(f(s^*) = \frac{hQ}{\pi_1 a})</td>
<td>(\phi(k^*) = \frac{\sigma hQ}{\pi_1 a})</td>
</tr>
<tr>
<td>Charge per Unit Short ((\pi_2))</td>
<td>(\pi_2 E_s)</td>
<td>(F(s^*) = 1 - \frac{hQ}{\pi_2 a})</td>
<td>(\phi(k^*) = 1 - \frac{hQ}{\pi_2 a})</td>
</tr>
<tr>
<td>Charge per Unit Short per Unit Time ((\pi_3))</td>
<td>(\pi_3 T_s)</td>
<td>(E_s(s^*) = \frac{hQ}{\pi_3})</td>
<td>(G(k^*) = \frac{hQ}{\sigma \pi_3})</td>
</tr>
<tr>
<td>Charge per Unit of Lost Sales ((\pi_L))</td>
<td>(\pi_L E_s)</td>
<td>(F(s^*) = \frac{\pi_L a}{hQ + \pi_L a})</td>
<td>(\phi(k^*) = \frac{\pi_L a}{hQ + \pi_L a})</td>
</tr>
</tbody>
</table>
Determination of the Order Quantity

All our examples have determined the reorder point given the order quantity. The following examples illustrate the determination of the order quantity when the reorder point is given, and the determination of optimum values for both variables simultaneously.

Example 12: Optimum order quantity given the reorder point

We continue from Example 9 in which the shortage cost is $200 per unit short. The demand during the lead time is Normal with $\mu = 25$ and $\sigma = 10$. If the reorder point is fixed at 50, what is the optimum order quantity?

For a Normal distribution the expected shortage cost is

$$C_s = \pi_2 \sigma G(k_s)$$

where $k_s = (s - \mu)/\sigma$. For $s = 50$ and $k_s = 2.5$,

$$G(2.5) = 0.0020, \quad E_s = 0.020, \quad C_s = 4.$$  

Then the optimum order quantity is

$$Q^* = \sqrt{\frac{2a(K + C_s)}{h}} = \sqrt{\frac{2(100)(800 + 4)}{10}} = 126.8$$

or 127 (conservatively rounded up).

Example 13: Both optimum order quantity and reorder point

In the previous examples we fixed one of the decisions and found the optimum value of the other. We need an iterative procedure to find both, $Q^*$ and $s^*$. We use the expression below sequentially.

$$Q = \sqrt{\frac{2a(K + C_s)}{h}} \cdot \phi(k_s) = 1 - \frac{hQ}{\pi_2a} \cdot C_s = \pi_2 \sigma G(k_s)$$

First assume $C_s = 0$ and find the optimum order quantity,

$$Q = 126.5.$$  

Using this value of $Q$, we find the optimum reorder point

$$k_s = 1.53 \text{ or } s = 40.3.$$  

The expected shortage per period with this reorder point is

$$C_s = \pi_2 \sigma G(1.53) = (200)(10)(0.02736) = 54.72$$

For this value of $C_s$
\[ Q = 130.7. \]

Using this value of \( Q \), we find the optimum reorder point

\[ k_s = 1.51 \text{ or } s = 40.1. \]

Computing the associated \( C_s \) we find

\[ Q = 130.9. \]

It appears that the values are converging, so we adopt the policy

\[ Q^* = 131 \text{ and } s^* = 40. \]