Controls Exam

Problem 1 (20 points):

Match the maps of the pole-zero plots with the correct time domain responses. Please note that some maps cannot be matched with any response. You need to explain your answers for full credit!!!

Map 1

Map 2

Map 3

Map 4

Map 5

Map 6

Response 1

Response 2

Response 3

Response 4
Explanation for Problem 1

Match for Response 1:

Match for Response 2:

Match for Response 3:

Match for Response 4:
Problem #2: (40 points)

Calculate the complete solution to:

\[ \ddot{y} + 4 \dot{y} + 4y = 4t^2 \]

for initial conditions \( y(0) = 1 \) and \( \dot{y}(0) = 0 \), using the method of undetermined coefficients (Hint: see the attached tables).
Problem #3: (40 points)

In this problem you will develop a closed loop model for position control of a mechanical system. A schematic of the system is attached (neglect the inertia of the pinion).

a. Derive a differential equation model for the system.

b. Compute the plant transfer function, which relates the imposed torque \((T(t))\) to the displacement \((x)\).

c. We would like to control the position \(x\). Draw a block diagram for the closed loop system, assuming: perfect measurement of \(x\), a reference input \((x_{IN})\), and a PD control scheme.

d. Compute the closed loop transfer function which relates \(x\) and \(x_{IN}\), as a ratio of polynomials in the operator \(D\) (defined as \(\frac{d}{dt}\)).

e. Compute the natural frequency and viscous damping factor for the closed loop system.

f. Assume that the closed loop system is underdamped, and sketch the response to a step input \(x_{IN}\).

g. Sketch and explain physically how the step response changes as you vary each of the two control gains.
### Table 2.1

Differential equation: \( ay'' + by' + cy = 0 \) or \( (aD^2 + bD + c)y = 0 \)
Characteristic equation: \( am^2 + bm + c = 0 \) or \( aD^2 + bD + c = 0 \)

<table>
<thead>
<tr>
<th>Nature of the roots of the characteristic equation</th>
<th>Condition on the coefficients of the characteristic equation</th>
<th>Complete solution of the differential equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real and unequal ( m_1 \neq m_2 )</td>
<td>( b^2 - 4ac &gt; 0 )</td>
<td>( y = c_1e^{m_1x} + c_2e^{m_2x} )</td>
</tr>
<tr>
<td>Real and equal ( m_1 = m_2 )</td>
<td>( b^2 - 4ac = 0 )</td>
<td>( y = c_1e^{m_1x} + c_2xe^{m_1x} )</td>
</tr>
<tr>
<td>Conjugate complex ( m_1 = p + iq ) ( m_2 = p - iq )</td>
<td>( b^2 - 4ac &lt; 0 )</td>
<td>( y = e^{qx}(A \cos qx + B \sin qx) )</td>
</tr>
</tbody>
</table>

### Table 2.2

Differential equation: \( ay'' + by' + cy = f(x) \) or \( (aD^2 + bD + c)y = f(x) \)

<table>
<thead>
<tr>
<th>( f(x) )†</th>
<th>Necessary choice for the trial particular integral ( Y )‡</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( a )</td>
<td>( A )</td>
</tr>
<tr>
<td>2. ( ax^n ) (( n ) a positive integer)</td>
<td>( A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n )</td>
</tr>
<tr>
<td>3. ( ae^{ux} ) (( r ) either real or complex)</td>
<td>( Ae^{ux} )</td>
</tr>
<tr>
<td>4. ( \alpha \cos kx )§</td>
<td>( A \cos kx + B \sin kx )</td>
</tr>
<tr>
<td>5. ( \alpha \sin kx )</td>
<td></td>
</tr>
<tr>
<td>6. ( ax^n e^{ux} \cos kx )</td>
<td>( (A_0x^n + \cdots + A_{n-1}x + A_n)e^{ux} \cos kx ) + ( (B_0x^n + \cdots + B_{n-1}x + B_n)e^{ux} \sin kx )</td>
</tr>
<tr>
<td>7. ( ax^n e^{ux} \sin kx )</td>
<td></td>
</tr>
</tbody>
</table>

† When \( f(x) \) consists of a sum of several terms, the appropriate choice for \( Y \) is the sum of the \( Y \) expressions corresponding to these terms individually.

‡ Whenever a term in any of the \( Y \)'s listed in this column duplicates a term in the complementary function, all terms in that \( Y \) expression must be multiplied by the lowest positive integral power of \( x \) sufficient to eliminate all such duplications.

§ The hyperbolic functions \( \cosh kx \) and \( \sinh kx \) can be handled either by expressing them in terms of exponentials or by using formulas entirely analogous to those in lines 4 to 7.
Problem 43
Problem 1: The figure below shows a powered lawn mower with rear wheels 2 driven while the front wheels 1 roll without power. The mower may also be pushed by the handle at 3. The center of gravity in the plane of motion is indicated by G, and it is located by the distances from the front and rear axles, $L_1$ and $L_2$, respectively, with $L_t = L_1 + L_2$, and sits at a height, $h$, from ground level. Assume the total mass of the mower is $m_v$, and the moment of inertia about the center of gravity is $I_G$.

(a) It is observed that on start-up the rear wheels spin as the mower accelerates. Assume that the static and dynamic coefficients of friction between the rear tires and the ground are $\mu_s$ and $\mu_k$, respectively. Draw a free-body diagram and derive expressions for the contact forces at the front and rear axles, $N_f$ and $N_r$, respectively. The forward direction should be designated positive $x$ to the right and vertical is positive $z$ upward.

(b) First, determine an analytical expression for the start-up forward acceleration, $a_x$, in terms of parameters provided. Assume that $P = 0$. Then, given $h = 215$ mm, $L_t = 500$ mm, $L_2 = 200$ mm, $m_v = 50$ kg, $m_s = 0.7$, and $m_k = 0.5$, calculate a numerical value for $a_x$.

(c) It is more realistic to include a total resistance force acting on the mower as it moves over grass being mowed. Assume a net linear ‘rolling resistance’ force is applied horizontally at G, and $P = 0$. Derive a differential equation for the forward velocity, $v_x$, assuming a rolling resistance modeled by $F_r = B_r \cdot v_x$ is included.

(d) Explain how you would estimate a value for $B_r$ for a given lawn mower.
Problem 2: The system shown below is a simplified representation of a positioning mechanism comprised of a rigid link having moment of inertia, $J$, about a fixed pivot, restrained by rotational spring and damper elements. These elements have stiffness, $K$, and damping coefficient, $B$. Assume the range of motion for $\theta$ is small, forced by a linear motion actuator connected to the link at point A. This is an electromechanical actuator, and the drive voltage $v(t)$, induces current $i_a$ in a series armature circuit which has resistance, $R_a$, and inductance, $L_a$. Assume damping forces on the actuator drive rod can be estimated by the coefficient, $b$, and that the translational mass of the drive rod is accounted for in the rigid link inertia. The manufacturer specifies that the actuator electromechanical force is related to the current by, $F_{em} = r \cdot i_a$.

![Diagram of the system](image)

(a) Develop a model of this system, identify the independent state variables. (b) Derive state equations for this system and write in state-space form.

(c) If the mass of the drive rod is $m_r$, derive an expression for the ‘effective rigid link inertia’ that includes this mass effect.

(d) It has been found that the state equations are given by:

$$
\begin{bmatrix}
\dot{h} \\ \dot{\lambda} \\ \dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
-24 & 300 & -288 \\ -3 & -500 & 0 \\ 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
h \\ \lambda \\ \theta
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\ 1 \\ 0
\end{bmatrix} v(t)
$$

Use these equations to derive the system characteristic equation.

(e) For the equations in part (d), find the system eigenvalues. Show all steps in your work. Use attached ‘Useful Formulas’, if necessary.

(f) If the link/spring/damper system is now isolated from the actuator, and an ideal input force is applied at point A, the system can be modeled as a second order system. Write or derive the differential equation for this system in terms of $\theta$, and derive the expressions for the natural frequency, $\omega_n$, and damping ratio, $\zeta$, in terms of system parameters.

(g) For the case in (f), show that the transfer function between the angular position, $\theta$, and the input force, $F(t)$, takes the form, $G(s) = C\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, and express $C$ in terms of system parameters.
Useful Formulas

**ALGEBRA**

**QUADRATIC EQUATIONS**

Any quadratic equation may be reduced to the form,

\[ ax^2 + bx + c = 0 \]

Then

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

If \( a, b, \) and \( c \) are real then:
- If \( b^2 - 4ac \) is positive, the roots are real and unequal;
- If \( b^2 - 4ac \) is zero, the roots are real and equal;
- If \( b^2 - 4ac \) is negative, the roots are imaginary and unequal.

**CUBIC EQUATIONS**

A cubic equation, \( y^3 + py^2 + qy + r = 0 \) may be reduced to the form,

\[ x^3 + ax + b = 0 \]

by substituting for \( y \) the value, \( x - \frac{p}{3} \). Here

\[ a = \frac{1}{3} (3q - p^2) \text{ and } b = \frac{1}{2} (2p^3 - 9pq + 27r). \]

For solution let,

\[ A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \quad B = -\sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \]

then the values of \( x \) will be given by,

\[ x = A + B, \quad \frac{A + B}{2} + \frac{A - B}{2} \sqrt{-3}, \quad \frac{A + B}{2} - \frac{A - B}{2} \sqrt{-3}. \]

If \( p, q, r \) are real, then:
- If \( \frac{b^2}{4} + \frac{a^3}{27} > 0 \), there will be one real root and two conjugate complex roots;
- If \( \frac{b^2}{4} + \frac{a^3}{27} = 0 \), there will be three real roots of which at least two are equal;
- If \( \frac{b^2}{4} + \frac{a^3}{27} < 0 \), there will be three real and unequal roots.

From CRC Standard Math Tables
27th edition, CRC Press
UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MECHANICAL ENGINEERING
Fall 2013 Design Methods Qualifying Exam

ACADEMIC TEST MATERIAL

Exam Information:
• This is a timed exam. Please carefully observe the time limit of 60 minutes.
• Important: please print name on each page. (Pages will be separated during grading.)
• Allowed: this exam paper and accompanying handouts.
• Add pages of additional work as needed. Please clearly print your name on each additional sheet.
• Problems consist of 100 possible points.

Please sign below, in agreement with the following1:
• I have not received information regarding the contents of this exam from other students.
• I will not exchange information about this exam with any other students before they have completed and turned in their exam.

Name (please print): ____________________________ Signature: ____________________________

1 If you can not sign in good conscience, leave the signature blank and make arrangements to speak with the proctor.
When completing items 1-2, refer to the sample product illustrated on the last page of the exam. Assume that the child and the parent steer and propel the tricycle simultaneously. Also, note that only the front wheel rotates to steer (i.e., both the child's steering handle and the parent's steering handle rotate the front wheel).

1. Draft a list of customer requirements for the tricycle.
2. Develop a black box model of the tricycle for the activity described above. The system boundary should encompass the tricycle itself, with the child and parent interacting with the system and providing input.
3. Develop a function structure for the tricycle. (Don’t forget to print your name on each page if you use additional sheets.) Consult the attached table of functional basis terms, as needed.
4. Describe the advantages and limitations of using a Pugh Chart for concept selection.
Pictures from walmart.com and amazon.com

Product Tour

1. Removable Push & Steer Handle
2. High-back saddle with 3-point seat belt
3. Front Basket
4. Rear gravity-lock dump bin
5. Folding foot rest

Click a feature from the list or a number in the picture to view details

Product Description from Amazon.com

From the Manufacturer
The Schwinn Easy Steer trike is made with a heavy duty trike frame with a patented concealed steering system. This removable steering system will allow the rider to pedal and steer the tricycle or the parent can take control and steer with the push/steer handle. The high back saddle with 3 point seat belt allows young riders to feel secure while adult controls the action. Wide EVA tires on enclosed wheels are stylish and long lasting.

Product Description
Every child remembers their first bike. Make those special memories with the Schwinn Easy Steer Tricycle! Great features such as the trike's high back seat and 3-point seat belt, provide the perfect combination of comfort and child safety. The Schwinn Easy Steer Trike makes learning to ride fun and easy, while the wide, long-lasting EVA tires offer a stylish look your child is sure to love. With a removable push/steering handle, adults are able to walk comfortably as they control the steering and speed of the tricycle. Gender Unisex Color White and Red Frame Steel Ages 2-4 years Assembly Assembly Required Removable push/steer handlebars High back saddle with 3 point seat belt Wide EVA tires on enclosed wheels Front and rear bins for toys or supplies Length 26 in. Width 20.5 in. Height 21.5 in.
<table>
<thead>
<tr>
<th>Class</th>
<th>Basic</th>
<th>Flow class restricted</th>
<th>Synonyms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel</td>
<td>Import</td>
<td>Input, Receive, Allow, Form Entrance, Capture</td>
<td>Discharge, Eject, Dispose, Remove</td>
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<td></td>
<td>Export</td>
<td>Transport (M)</td>
<td>Lift, Move</td>
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<td></td>
<td>Transfer</td>
<td>Transport (E)</td>
<td>Translate</td>
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<td></td>
<td>Guide</td>
<td>Rotate</td>
<td>Turn, Spin</td>
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<td>Allow DOF</td>
<td>Constram, Unlock</td>
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<td>Support</td>
<td>Stop</td>
<td>Insulate, Protect, Prevent, Shield, Inhibit</td>
<td>Steady</td>
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<td>Secure</td>
<td>Attach, Mount, Lock, Fasten, Hold</td>
<td>Orient, Align, Locate</td>
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<td>Connect</td>
<td>Couple</td>
<td>Join, Assemble, Attach</td>
<td>Combine, Blend, Add, Pack, Coalesce</td>
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<td>Mix</td>
<td>Separate</td>
<td>Switch, Divide, Release, Detach, Disassemble, Dissubct, Valve</td>
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<td>Remove (M)</td>
<td>Cut, Polish, Sand, Drill, Lathe</td>
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<td>Refine</td>
<td>Purify, Strain, Filter, Percolate, Clear</td>
<td>Diverge, Scatter, Disperse, Diffuse, Empty</td>
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<td>Fill, Provide, Replenish, Expose</td>
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<td>Control magnitude</td>
<td>Actuate</td>
<td>Actuate</td>
<td>Start, Initiate</td>
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<td>Control, Allow, Prevent, Enable/Disable, Limit, Interrupt</td>
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<td>Measure</td>
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<td>Calculate</td>
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</table>

Note: Repeated synonyms are italicized.
Problem 1 (20 points in total)

Briefly answer the following questions (4 points each)

a) Explain the differences between orthogonal cutting and oblique cutting in terms of the definition, cutting force, and chip shape.

b) In sheet metal shearing processes, sometimes the shear blade is designed with a shear angle. What is the purpose of this shear angle?

c) Dead metal zones can occur in both metal extrusion and machining processes. Sketch to show where these dead metal zones can occur. Comment on the effects of the dead metal zones in each of these two processes.

d) Sketch to show the difference between down milling and up milling. Compare these two milling methods in terms of the type of materials they are generally used for, cutting force, and surface finish.

e) Explain the statistically background of the 6σ methodology, which allows 3.4 parts/million defect rate. What is the Cp of a 6σ process? What is the Cpk of a 6σ process?
Problem 2 (25 points in total)

Multiple choice problems (questions 1-6 carry 3 points each, while question 7 carries 2 points; there is only one correct answer)

(1) Which of the following should NOT be considered as an indicator of good machineability using automated machinery?
   a. Low cutting energy
   b. Continuous chip formation
   c. Good surface finish
   d. Minimum tool wear

(2) Which of the following can NOT be performed on a conventional lathe (e.g., a lathe in our machine shop)?
   a. Taper turning
   b. Threading
   c. Hole drilling along the axial direction of a part
   d. Hole drilling along the radial direction of a part

(3) Which of the following processes use mechanical energy source in machining?
   a. Electrochemical grinding
   b. Laser beam machining
   c. Ultrasonic machining
   d. Wire EDM

(4) Which of the following processes would be appropriate to drill a blind hole with a square cross section, 0.25 in on the side and 1 in deep in a steel workpiece?
   a. abrasive water jet machining
   b. chemical milling
   c. laser beam machining
   d. wire EDM

(5) Degree of polymerization is which of the following?
   a. Average number of mers in the molecule chain
   b. Proportion of the monomer that has been polymerized
   c. Sum of the molecular weights of the mers in the molecule
   d. None of the above

(6) Which of the following most closely typifies the sintering temperature in powder metallurgy processes?
   a. 0.5 Tm  b. 0.8Tm  c. 1.1 Tm  d. 0.3 Tm

(7) Polymer is either amorphous or crystalline. Is this statement true or false?
   a. True  b. False
Problem 3 (25 points in total)

Identify the following manufacturing processes (questions a-d carry 2 points each, while questions e-h carry 3 points each):

(a) Extruded product
    Ram
    Dummy block
    Die

(b) Die

(c) Fluid

(d) Punch
    Extruded tube

(e) Wire or rod
    Die

(f) Thrashing
    Male die

(g) Primary motion (– C)
    Transient surface
    Machined surface
    Continuous feed motion (– Z)
    Tool

(h) Pressurized Resin
    Air
    preform
Problem 4 (25 points in total)

Forces involved in metal cutting can be estimated both experimentally and theoretically. Experimentally, the material removal rate and specific cutting energy are used. Theoretically, the ideal orthogonal cutting model is used. Consider a continuous turning process that is close to the ideal orthogonal cutting condition. The workpiece is a stainless steel bar with an initial diameter of 100 mm. The rough cut is to be made at a feed rate \( f_s = 0.3 \) mm/rev and a depth of cut \( d = 0.5 \) mm. The spindle rotates at \( N = 500 \) rpm. The rake angle of the tool is 10°. The thickness of the deformed chip is 0.45 mm. The specific cutting energy for this material is 2.3 J/mm\(^3\) for undeformed chip thickness of 1 mm. For other undeformed chip thicknesses, the specific energy can be calculated using \( E = E_i h^{-a} \), where \( E_i \) is the specific energy when the undeformed chip thickness is 1 mm, \( h \) is the undeformed chip thickness in the cutting process, and \( a \) is an empirical coefficient that can be taken as 0.3 for stainless steel.

Based on the above information,

1. Calculate the material removal rate of the turning process.
2. Estimate the cutting force based on the given specific cutting energy, assuming the efficiency of the machine is 75%.
3. Sketch a picture of the corresponding ideal orthogonal cutting model and indicate the key parameters including the rake angle, undeformed chip thickness, deformed chip thickness, and cutting speed.
4. Find the shear angle.
5. If the shear strength of the material is 250 MPa, calculate the shear force on the shear plane (if you could not find the shear angle from (4), you can assume that the shear angle is 30°).
Some useful formulae for Problem 4

• Power required for cutting = Modified spec. cutting energy * material removal rate

• Cutting Force [N] * Cutting Speed [m/s] = Power delivered by the machine [W]

• Power required for cutting = Power delivered by the machine * Efficiency

• \[\tan(\text{shear angle}) = \frac{(h/hc)\cos(\text{rake angle})}{1-(h/hc)\cos(\text{rake angle})}\]
  Where \(h\) is thickness of the undeformed chip and \(hc\) is thickness of the deformed chip

• Area of the shear plane = thickness of the undeformed chip * depth of cut / \(\sin(\text{shear angle})\)

• Shear Force = Shear strength * Area of the shear plane
Machine Elements Design
August 2013

Problem 1

A. Provide an example of an application of helical compression springs.

B. A helical compression spring is wound using a 2.5 mm diameter (d) music wire. The spring has an outside diameter of 30 mm with plain ground ends, and 16 total coils. Estimate the free length of the spring needed to ensure that if it is compressed solid the torsional stress does not exceed the yield strength. State your assumptions in calculating the number of active coils. Assume that the spring material has a yield strength of 900MPa. Assume that the spring index \( C = \frac{D}{d} \) where \( D \) is the mean spring diameter, \( K_B = \frac{(4C+2)}{(4C-3)} \), and the yield strength is given by the following equation:
\[
S_{sy} = \frac{8FK_BD}{(\pi d^3)}
\]

Problem 2

The figure on page 2 illustrates a micromachined sensor design taken from the literature\(^1\). Assume that this device is fabricated using an isotropic material with a bending modulus of \( E \), and a yield strength of \( S_{sy} \). Also assume that the system is loaded by a point vertical load \( (F) \) at the center of the sensor proof mass. Answer the following questions:

A. Qualitatively describe how you would compute the maximum deflection caused by the point load on the proof mass. Please provide the detailed steps required to solve the deflection problem, but do not set up the equations to obtain the explicit formula for deflection.

B. Qualitatively describe the process for identifying the location of the maximum stress in this device in the presence of the previously mentioned vertical load \( F \).

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This exam is closed book and closed notes.

1. Write your answers on a separate sheet of paper.

2. To obtain full credit, you should explain your reasoning for your answers to all questions.

3. Each question below is worth 25% of this part of the exam.

1. Let $X$ be a continuous random variable with distribution function $F_X(\cdot)$.
   (a) Show that the random variable $Y = F_X(X)$ has a Uniform(0,1) distribution.
   (b) Identify the step in your proof which fails if $X$ is not continuous and explain why it fails.
   (c) Provide a specific counterexample showing that the result does not work if $X$ is not continuous.

2. Let $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\lambda)$ with $X$ and $Y$ being independent. Recall the Poisson p.m.f.: $X \sim Pois(\lambda)$, $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$, for $k = 0, 1, 2, \ldots$.
   (a) Directly calculate the distribution of $X + Y$.
   (b) Calculate $P[X = k \mid X + Y = n]$.
   (c) Calculate $E[X \mid X + Y = n]$ and $\text{Var}[X \mid X + Y = n]$.

3. Suppose we break a stick of length $K$ at a (uniformly) random place. For each outcome $\omega$ we are interested in the ratio $S(\omega)/L(\omega)$, where $S(\omega)$ is length of the shorter piece, and $L(\omega)$ is the length of the longer piece.
   (a) Find the expected value of $S/L$.
   (b) Write an integral expression for the variance of $S/L$ (you do not need to evaluate the expression).
   (c) Without calculus, can you easily calculate $E[L/S]$? If so, give the answer. If not, give a bound using your answer to (a).

4. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables taking values on the real line. We define a random walk to be the infinite sequence
   \[ S_n = \sum_{i=1}^{n} X_i \quad n \geq 1. \]
(a) Suppose $0 < E X_1 < \infty$. Prove that $S_n \to \infty$ with probability one.

(b) Suppose we examine the sequence

$$\frac{S_n - nEX_1}{\sigma \sqrt{n}} \quad n \geq 1,$$

where $\sigma$ is the standard deviation of $X_1$. Under what conditions does the sequence converge (and in what sense does it converge)? Give the distribution function of the random variable to which the sequence converges.

(c) Suppose we stop the random walk at a random time $N$, with $N$ being independent of the sequence $\{X_i, i \geq 1\}$. Show directly (without Wald’s identity) that $E S_N = E X_1 \cdot E N$. What conditions are necessary on the $X_i$ and $N$ for your argument to be valid?
## Discrete Distributions

<table>
<thead>
<tr>
<th>Discrete distribution</th>
<th>pmf</th>
<th>$\varphi(t)$</th>
<th>$E[X]$</th>
<th>$Var[X]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli</td>
<td>$p$</td>
<td>$e^t p + 1 - p$</td>
<td>$p$</td>
<td>$p(1 - p)$</td>
</tr>
<tr>
<td>Binominal with parameters $n$</td>
<td>$\binom{n}{x} p^x (1 - p)^{n-x}$</td>
<td>$(pe^t + 1 - p)^n$</td>
<td>$np$</td>
<td>$np(1 - p)$</td>
</tr>
<tr>
<td>$0 \leq p \leq 1$</td>
<td>$x = 0, 1, ..., n$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Poisson with parameter $\lambda$</td>
<td>$e^{-\lambda} \frac{\lambda^x}{x!}$</td>
<td>$e^{\lambda(e^t - 1)}$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Geometric with parameter $p$</td>
<td>$p(1 - p)^{x-1}$</td>
<td>$\frac{pe^t}{1 - (1 - p)e^t}$</td>
<td>$p^{-1}$</td>
<td>$p^{-2}(1 - p)$</td>
</tr>
<tr>
<td>$0 \leq p \leq 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Negative binomial with parameters $r$ and $p$.</td>
<td>$\binom{n-1}{r-1} p^r (1 - p)^{n-r}$</td>
<td>$\left(\frac{pe^t}{1 - (1 - p)e^t}\right)^r$</td>
<td>$rp^{-1}$</td>
<td>$rp^{-2}(1 - p)$</td>
</tr>
<tr>
<td>$0 \leq p \leq 1$</td>
<td>$n = r, r + 1, ...$</td>
<td></td>
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</tbody>
</table>
Continuous Distributions

Discrete distribution
Uniform over \((a,b)\)

Exponential with parameter \(\lambda > 0\)

Gamma with parameters \((n,\lambda), \lambda > 0\)

Normal with parameters \((\mu,\sigma^2)\)

\[
\begin{align*}
\text{Uniform over } (a,b) & : & \begin{cases} 
0, & a < x < b \\
(b-a)^{-1}, & \text{ow}
\end{cases} \\
\text{Exponential } & : & \begin{cases} 
\lambda e^{-\lambda x}, & x \geq 0 \\
0, & x < 0
\end{cases} \\
\text{Gamma } & : & \begin{cases} 
\frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}, & x \geq 0 \\
0, & x < 0
\end{cases} \\
\text{Normal } & : & \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\end{align*}
\]
1. Write your answers on a separate sheet of paper.

2. To obtain full credit, you must explain your reasoning for your answers to all questions.

1. (33%) Consider the following linear program and note that it is a "max":

\[
\begin{align*}
\text{max} & \quad 10x_1 + 24x_2 + 20x_3 + 20x_4 + 25x_5 \\
\text{s.t.} & \quad x_1 + x_2 + 2x_3 + 3x_4 + 5x_5 \leq 19 \\
& \quad 2x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \leq 57 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{align*}
\]

(a) Find the dual and verify that \((\pi_1, \pi_2) = (4, 5)\) is a feasible solution to the dual.

(b) Use the information from (a) to derive an optimal solution to both the primal and dual.

2. (33%) Consider the following system of linear inequalities:

\[
\begin{align*}
Ax + By &= b \\
Cx + Dy &\geq d \\
x &\geq 0, y \leq 0.
\end{align*}
\] (1)

Here, \(A\) is \(m_1 \times n_x\), \(B\) is \(m_1 \times n_y\), \(C\) is \(m_2 \times n_x\) and \(D\) is \(m_2 \times n_y\), \(b\) is an \(m_1\)-vector and \(d\) is an \(m_2\) vector.

Formulate a theorem of the alternative for (1). That is, state another linear system (call it system (2)) such that either (1) has a solution or (2) has a solution but not both. Prove your claim.

3. (34%) Let \(\hat{x}\) be a feasible point for the following linear program:

\[
\begin{align*}
\text{min} & \quad cx \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0.
\end{align*}
\]

Let \(d\) be a given direction of movement, and consider the family of solutions \(x(t) = \hat{x} + td\) for scalar \(t \geq 0\).

(a) Derive in algebraic terms the maximum steplength, \(t^*\), that can be taken without violating feasibility of the linear program.

(b) Under what conditions will the linear program be unbounded?