1) \( f(x, t) \)

No point \( x^* \) such that \( f(t, x^*) \leq f(t, x) \quad \forall x \)

a) No solution because he is trying to minimize an upper semi continuous function.

\[ \min_x f(t-x) \] doesn't exist.

b) \( \min_x \mathbb{E}[f(T-x)] = \min_x \mathbb{E}\left[ \int_{-\infty}^{\infty} f(t-x) g(t) \, dt \right] \)

where \( g(t) \) is density function of \( t \)

\[ = \min_x \mathbb{E}\left[ \int_{T_1}^{\infty} g(t) \, dt + \int_{-\infty}^{T_1} (t-x) g(t) \, dt \right] \]

c) Solution should exist because integrals exist and are finite. In addition, the expected value is continuous and defined over a compact set \( \{ T_1 \leq x \leq T_2 \} \)

\[ \min_{T_1 \leq x \leq T_2} \mathbb{E}(x) - \mathbb{E}(x) + \int_{-\infty}^{\infty} T g(t) \, dt - x \left[ \mathbb{E}(x) - \mathbb{E}(x) \right] \]

\[ \mathbb{E}(x) = \int_{-\infty}^{\infty} g(t) \, dt \]

\[ \Theta(x) = \int_{-\infty}^{x} g(t) \, dt + \int_{x}^{\infty} g(t) \, dt = \int_{-\infty}^{\infty} g(t) \, dt \]

\[ \mathbb{E}(x) = \mathbb{E}(x) - \lambda \mathbb{E}(x) - \left[ \lambda \mathbb{E}(x) + \int_{-\infty}^{\infty} g(t) \, dt \right] \geq 0 \]
2. Intersection of family of closed sets is closed

Let $S = \bigcap S_\alpha$ with $S_\alpha$ closed for all $\alpha$

Let $x^k \to \bar{x}$ (where $\bar{x}$ is a limit point)

and assume $x^k \in S_\alpha$ \(\forall k \in \mathbb{N}\).

then all $x^k \in S_\alpha$ for any $\alpha$.

Now, because $S \subseteq S_\alpha$, it is also true that $\bar{x} \in S_\alpha$ for any $\alpha$.

Thus $\bar{x} \in S$ and $S$ is closed.

The conclusion is based on the following definition of a closed set.

**Def.** The set $S \subset \mathbb{R}^m$ is closed if, for any sequence of points $x^m \in S$ convergent to a limit point, the limit point belongs to $S$.

3. Let $S_2 = \{x : 0 \leq x \leq 1 - \frac{1}{2^3}\}$

Let $S = \bigcup S_\alpha$

Note $S_\alpha$ is closed for all $\alpha$.

There are an infinite number of sets $S_\alpha$ and their union is $\{x : 0 \leq x \leq 1\}$ which is not closed (nor open).
4. \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) continuous. Show that \( S = \{ x : f(x) \leq 0 \} \) is closed.

Let \( x^k \rightarrow x \) where \( x^k \in S \) and \( x \notin S \).

Then \( S \) is not closed and \( f(x) > 0 \).

From the definition of continuity of \( f \) for all \( \varepsilon > 0 \),

\( \exists \delta > 0 \) such that

\[ |f(x) - f(x^k)| < \varepsilon \quad \forall |x - x^k| < \delta \]

Let \( \varepsilon > 0 \) be such that

\[ f(x) - \varepsilon < f(x^k) \leq 0 \]

for all \( k \in \mathbb{N} \)

where \( N \) is large enough so that \( |x^k - x| < \delta \).

Thus for \( \varepsilon \) sufficiently small we have

\[ f(x^k) - \varepsilon > 0 \quad \text{or} \quad 0 < f(x^k) \leq 0 \]

contradiction!

5. Example where \( f \) discontinuous and

the set \( S = \{ x : f(x) \leq 0 \} \) not closed.

Let \( f(x) = \left\{ \begin{array}{ll} 1 & x = 1 \\ -1 & x \neq 1 \end{array} \right. \)

\[ f(x^k) \quad \xrightarrow{\text{convergence}} \quad x \]

The point \( x = 1 \) is not in the set \( S \) but is

\[ \text{continuation close to the set.} \]