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The characteristic landscape equation for an AR(2) landscape

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Abstract

Grover's equation [L.K. Grover, Local search and the local structure of some NP-complete problems, Operations Research Letters 12 (1992)] and, equivalently, the Laplacian equation [J.W. Barnes, S. Dokov, B. Dimova, A. Solomon, A theory of elementary landscapes, Applied Mathematics Letters 16 (2003)] define an elementary landscape which has favorable properties for direct search methods. Dimova et al. [B. Dimova, J.W. Barnes, E. Popova, Arbitrary elementary landscapes & AR(1) processes, Applied Mathematics Letters (in press)] prove that the autocorrelation function associated with an arbitrary elementary landscape is consistent with an AR(1) time series. In this work, we develop the characteristic landscape equation for AR(2) consistent landscapes and show that they also possess favorable properties for direct search methods.

Keywords: Elementary landscapes; Grover's equation; Laplacian equation; Autocorrelation function; Autoregressive processes

1. Introduction

Barnes et al. [2] define a landscape for a combinatorial optimization problem (COP) as $\mathcal{L} = (X, f, \mathcal{N})$, where $X = [x_i]$ is the finite solution space, $f = [f(x_i)] = [f_i]$ is the real objective function vector over all X, and $\mathcal{N}$ is the search neighborhood defined by a digraph where the nodes are the $x_i \in X$. The neighborhood digraph has an associated adjacency matrix $A = [a_{ij}]$ and transition matrix $T = [t_{ij}]$. The transition matrix for an m step neighborhood is shown to be T^m. For each $x_i \in X$, a nonzero a_{ij} designates x_j as a neighbor of x_i and t_{ij} gives the probability of moving to x_j in the next move. For our current purposes, $t_{ij} = a_{ij}/d_i$ where $d_i = \sum_{\forall j} a_{ij}$, the degree of node i.

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Kemeny and Snell [4] define $\pi = [\pi_i]$ to be the *steady state vector* associated with T and show that $\alpha = \pi' T f = \pi' f$ is the expected value of f_i for the discrete-time Markov chain defined by T. Define $f_\alpha = [f(x_i) - \alpha] = [f_{\alpha i}]$ to be the α-normalized objective function vector. Let $[z_t] = [f_{i,t} - \alpha] = [f_{\alpha(i,t)}]$ be the time series yielded from a random walk on $\mathcal{L}$, as defined in [2] starting at x_i; i.e., the t-th normalized solution visited in the generation of the time series has objective function $z_t = f_{\alpha(i,t)}$. An autoregressive process of order 2, AR(2) (see [5] for more details), is defined by the recurrence equation

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t \tag{1}$$

where a_t is a random deviate with mean 0. This follows directly from the fact that $E(z_t) = 0$ for all t and $a_t = z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2}$.

Consider the temporally adjacent triplet, z_t, z_{t-1} and z_{t-2}. By definition, $z_{t-2} \equiv f_{\alpha i}$ for *some* x_i. For such specified z_{t-2}, the d_i neighbors of x_i are the $x_j \in \mathcal{N}_i$ with values $z_{t-1,j}$. For a given $x_j \in \mathcal{N}_i$, the d_j neighbors are the $x_k \in \mathcal{N}_j$ with values $z_{t,k} \equiv f_{\alpha,k}$.

The remainder of the work is divided into four sections. In Section 2, we develop a *characteristic landscape equation* for an AR(2) process and prove that any landscape that satisfies the characteristic equation will possess an autocorrelation function consistent with an AR(2) process. In Section 3, we investigate some properties of a landscape consistent with an AR(2) process. In Section 4, we develop a *characteristic landscape equation* for an AR(p) process. Finally, in Section 5, we state some conclusions and indicate future research directions.

2. The characteristic landscape equation for AR(2) process

Grover [1] defined elementary landscapes and showed that such landscapes have properties favorable for local search. Stadler [6] showed that if T is symmetric-regular, then $\mathcal{L}$ is elementary if and *only* if a univariate time series generated from a random walk on $\mathcal{L}$ is consistent with an autoregressive process of order 1, i.e., an AR(1) process. Dimova et al. [3] showed that *all* elementary landscapes are consistent with an AR(1) process.

Proposition 1. *If the time series based on a random walk on the landscape is consistent with an* AR(2) *process, then the landscape satisfies the characteristic landscape equation,* $(T^2 - \phi_1 T) f_\alpha = \phi_2 f_\alpha$.

Proof. The proof proceeds in three steps:

1. first, the one-step neighborhood of a specific neighbor of solution x_j which is a specific neighbor of x_i is considered;
2. next, the results are expanded to consider all neighbors, $x_j \in \mathcal{N}_i$; and
3. third, all possible starting solutions, x_i, are considered.

Step 1: We are given a specific solution x_i, with associated objective function value $z_{t-2} \equiv f_{\alpha i}$. Consider a specific one-step neighbor of x_i, $x_j \in \mathcal{N}_i$, with associated objective function value $z_{t-1,j} \equiv f_{\alpha,j}$. The cardinality of $\mathcal{N}_j$ is d_j. With x_i and x_j fixed, we average Eq. (1) over all d_j neighbors of x_j which yields (reading $k \in \mathcal{N}_j$ as "k such that $x_k \in \mathcal{N}_j$")

$$\sum_{k\in\mathcal{N}_j} \frac{z_{t,k}}{d_j} = \phi_1 \sum_{k\in\mathcal{N}_j} \frac{z_{t-1,j}}{d_j} + \phi_2 \sum_{k\in\mathcal{N}_j} \frac{z_{t-2}}{d_j} + \sum_{k\in\mathcal{N}_j} \frac{a_{t,k}}{d_j}. \tag{2}$$

Observe that

$$\sum_{k\in\mathcal{N}_j}\frac{z_{t,k}}{d_j}=\operatorname*{Avg}_{k\in\mathcal{N}_j} f_{\alpha,k}=T_j\, f_\alpha,\qquad \sum_{k\in\mathcal{N}_j}\frac{z_{t-1,j}}{d_j}=f_{\alpha,j}\qquad\text{and}\qquad \sum_{k\in\mathcal{N}_j}\frac{z_{t-2}}{d_j}=f_{\alpha,i}$$

where T_j is the j-th row of T. Substituting this into Eq. (2) we obtain

$$T_j\, f_\alpha=\phi_1 f_{\alpha,j}+\phi_2 f_{\alpha,i}+\sum_{k\in\mathcal{N}_j}\frac{a_{t,k}}{d_j},\qquad i=1,\dots,|X|,\ j: x_j\in\mathcal{N}_i. \tag{3}$$

Step 2: Taking the average of Eq. (3) over the d_i neighbors of x_i, i.e., over the $x_j\in\mathcal{N}_i$, we obtain

$$\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}T_j\, f_\alpha=\phi_1\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}f_{\alpha,j}+\phi_2\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}f_{\alpha,i}+\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}\sum_{k\in\mathcal{N}_j}\frac{a_{t,k,j}}{d_j},\qquad i=1,\dots,|X|. \tag{4}$$

Observing that $\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}T_j\, f_\alpha = T_i^2\, f_\alpha$, $\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}f_{\alpha,j} = T_i\, f_\alpha$, and $\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}f_{\alpha,i} = f_{\alpha,i}$, and substituting into Eq. (4), yields

$$T_i^2\, f_\alpha=\phi_1 T_i\, f_\alpha+\phi_2 f_{\alpha,i}+\frac{1}{d_i}\sum_{j\in\mathcal{N}_i}\sum_{k\in\mathcal{N}_j}\frac{a_{t,k,j}}{d_j}. \tag{5}$$

Taking the expectation of Eq. (5) yields

$$T_i^2\, f_\alpha=\phi_1 T_i\, f_\alpha+\phi_2 f_{\alpha,i},\qquad i=1,\dots,|X|. \tag{6}$$

Step 3: In matrix form, Eq. (6) can be written as $T^2 f_\alpha=\phi_1 T f_\alpha+\phi_2 f_\alpha$ or

$$(T^2-\phi_1 T)\, f_\alpha=\phi_2\, f_\alpha.\quad\square \tag{7}$$

Eq. (7) is the *characteristic landscape equation* for an AR(2) process. If $\phi_2=0$, Eq. (7) degenerates to the classical Laplacian equation for an AR(1) process, i.e., if $\phi_2=0$ and if T is invertible. As shown in [2], Eq. (7) may be expressed as $T f_\alpha=\phi_1 f_\alpha$. If $\mathcal{G}=T^2-\phi_1 T$ is a stochastic matrix, then an equivalent classical elementary landscape is present with associated Laplacian equation $\mathcal{G} f_\alpha=\phi_2 f_\alpha$.

Weinberger [7] defines the sample autocorrelation function of a time series, $[f_{\alpha i}]$, of length n generated by a random walk on $\mathcal{L}$. Dimova et al. [3] show that the matrix form of the theoretical autocorrelation function for *any* $\mathcal{L}$ is

$$\rho_\pi(s)=\frac{f_\alpha'\,\Pi\, T^s\, f_\alpha}{f_\alpha'\,\Pi\, f_\alpha} \tag{8}$$

where Π is a diagonal matrix with $\Pi_{ii}=\pi_i$ and $\pi=[\pi_i]$ is the steady state vector (see [8] for details) associated with T.

Proposition 2. *If the landscape satisfies the equation* $(T^2-\phi_1 T)\, f_\alpha=\phi_2\, f_\alpha$*, then the time series based on a random walk on this landscape is consistent with an* AR(2) *process.*

Proof. Using the Box and Jenkins [5] approach to analyze a time series, we will prove that behavior of the theoretical autocorrelation function $\rho_\pi(s)$ of such a time series is consistent with the autocorrelation function of an AR(2) process.

Eq. (7) can be written in the form

$$T^2 f_\alpha=\phi_1 T f_\alpha+\phi_2 f_\alpha. \tag{9}$$

Further, following Kemeny and Snell [4], we note that $T^0 = I$, which yields

$$\rho_\pi(0) = \frac{f'_\alpha \Pi T^0 f_\alpha}{f'_\alpha \Pi f_\alpha} = 1.$$

Premultiplying Eq. (8) by T^{-1} and substituting yields

$$\rho_\pi(1) = \frac{f'_\alpha \Pi T f_\alpha}{f'_\alpha \Pi f_\alpha} = \frac{f'_\alpha \Pi \phi_1 f_\alpha + f'_\alpha \Pi \phi_2 T^{-1} f_\alpha}{f'_\alpha \Pi f_\alpha} = \phi_1 \rho_\pi(0) + \phi_2 \rho_\pi(-1). \tag{10}$$

Similarly,

$$\rho_\pi(2) = \frac{f'_\alpha \Pi T^2 f_\alpha}{f'_\alpha \Pi f_\alpha} = \frac{f'_\alpha \Pi (\phi_1 T f_\alpha + \phi_2 f_\alpha)}{f'_\alpha \Pi f_\alpha} = \phi_1 \rho_\pi(1) + \phi_2 \rho_\pi(0). \tag{11}$$

The general relation is obtained by premultiplying Eq. (9) by T^{n-2} and substituting $T^n f_\alpha$ into Eq. (8). This yields the recurrent formula for the autocorrelation function of such landscape,

$$\rho_\pi(n) = \phi_1 \rho_\pi(n-1) + \phi_2 \rho_\pi(n-2).$$

Observing that $\rho_\pi(-n) = \rho_\pi(n)$, the simultaneous solution of Eqs. (10) and (11) yields

$$\rho_\pi(1) = \frac{\phi_1}{1-\phi_2} \qquad \text{and} \qquad \rho_\pi(2) = \phi_2 + \frac{\phi_1^2}{1-\phi_2}$$

which corresponds to the classical autocorrelation function of an AR(2) process [5]. □

3. Properties of an AR(2) landscape

This section investigates the structure of an AR(2) landscape from the perspective of local minima using Eq. (7). Depending on the values of the coefficients ϕ_1 and ϕ_2, an AR(2) process can be:

(a) stationary — the roots of its characteristic equation lie outside the unit circle; this is equivalent to requiring that $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, $-1 < \phi_2 < 1$,
(b) non-stationary — the roots of its characteristic equation lie on the unit circle, and
(c) explosive — the roots of its characteristic equation lie inside the unit circle.

In the context of this work, cases (b) and (c) cannot occur because X is finite and the directly related finite state Markov chain is used to generate the associated time series.

3.1. An upper bound for local minima for the two-step neighborhood

Consider a landscape $\mathcal{L}$ consistent with an AR(2) process. For any two-step neighborhood local minimum, x_i^{**},

$$T_i^2 f_\alpha - \phi_1 T_i f_\alpha = \phi_2 f_{\alpha i}^{**}. \tag{12}$$

The average of all two-step neighbors of x_i^{**}, $\text{Avg}_{k\in\mathcal{N}^2} f_{\alpha,k}$ equals $T_i^2 f_\alpha$. Since $f_{\alpha i}^{**}$ is less than or equal to the objective function value for any two-step neighbor,

$$f_{\alpha i}^{**} \le T_i^2 f_\alpha. \tag{13}$$

Eqs. (12) and (13) imply that $f_{\alpha,i}^{**} - \phi_1 T_i f_\alpha \le \phi_2 f_{\alpha,i}^{**}$ which yields

$$(1-\phi_2) f_{\alpha,i}^{**} \le \phi_1 T_i f_\alpha.$$

Since $1 - \phi_2 > 0$ (i.e. the process is stationary),

$$f_{\alpha i}^{**} \leq \frac{\phi_1}{1-\phi_2} T_i f_\alpha. \tag{14}$$

Hence $f_{\alpha i}^{**}$ is bounded above by $\rho_\pi(1) T_i f_\alpha$ which is equivalent to the autocorrelation at lag 1 times the average objective function value for the one-step neighbors of x_i^{**}.

3.2. *An upper bound for local minima for the one-step neighborhood*

For an AR(2) landscape, the set of all one-step local minima, the x_i^*, fall into two subsets: M_2 where the x_i^* are *also* two-step local minima and M_1 where the x_i^* are *not* two-step local minima.

We first consider M_2. Any $x_i^* \in M_2$ satisfies Eq. (12) and the following relations also hold:

$$f_{\alpha i}^* \leq T_i f_\alpha \tag{15}$$

$$f_{\alpha i}^* \leq T_i^2 f_\alpha. \tag{16}$$

By Eq. (14), $f_{\alpha i}^* \leq \frac{\phi_1}{1-\phi_2} T_i f_\alpha$.

We must consider two cases: $\phi_1 \geq 0$ and $\phi_1 \leq 0$.

- If $\phi_1 \geq 0$, multiplying Eq. (15) by $-\phi_1$ and adding $T_i^2 f_\alpha$ to both sides yields $T_i^2 f_\alpha - \phi_1 f_{\alpha i}^* \geq T_i^2 f_\alpha - \phi_1 T_i f_\alpha$. Since, by Eq. (12), $T_i^2 f_\alpha - \phi_1 T_i f_\alpha = \phi_2 f_{\alpha i}^*$, this implies $(\phi_1 + \phi_2) f_{\alpha i}^* \leq T_i^2 f_\alpha$.

  This leads to the following two subcases where we assume that $T_i^2 f_\alpha \leq 0$ (if $T_i^2 f_\alpha > 0$, there appear to be no additional meaningful conclusions that can be drawn):

  (i) If $\phi_1 + \phi_2 > 0$ and $T_i^2 f_\alpha \leq 0$, $f_{\alpha,i}^* \leq 0$ and $f_i^* \leq \alpha$, i.e., arbitrarily poor local optima of class M_2 cannot exist.

  (ii) If $\phi_1 + \phi_2 < 0$ and $T_i^2 f_\alpha \leq 0$, $f_{\alpha,i}^* \geq 0$. However, in this case, Eq. (16) requires that $f_{\alpha,i}^* \leq 0$. Therefore $f_{\alpha,i}^* = 0$ or $f_i^* = \alpha$. This implies that all $x_i \in M_2$ have $f_i^* = \alpha$.
- If $\phi_1 \leq 0$, multiplying Eq. (15) by $\frac{\phi_1}{1-\phi_2}$ yields $\frac{\phi_1}{1-\phi_2} T_i f_\alpha \leq \frac{\phi_1}{1-\phi_2} f_{\alpha i}^*$. This result joined with Eq. (14) implies $f_{\alpha i}^* \leq \frac{\phi_1}{1-\phi_2} f_{\alpha i}^*$ which directly yields $(1-\phi_1-\phi_2) f_{\alpha,i}^* \leq 0$. Since $(1-\phi_1-\phi_2) > 0$, $f_{\alpha,i}^* \leq 0$ and $f_i^* \leq \alpha$, i.e., arbitrarily poor local optima of class M_2 cannot exist.

Let us now consider M_1. Any $x_i^* \in M_1$ with value $f_{\alpha i}^*$ satisfies Eqs. (9) and (15).

- If $\phi_1 \geq 0$, the analysis is identical to that for the case where $x_i^* \in M_2$ and the same conclusion is reached.
- If $\phi_1 \leq 0$ *and* $\phi_1 + \phi_2 < 0$, multiplying Eq. (15) by $-\phi_1$ and adding $T_i^2 f_\alpha$ to both sides yields $T_i^2 f_\alpha - \phi_1 f_{\alpha i}^* \leq T_i^2 f_\alpha - \phi_1 T_i f_\alpha = \phi_2 f_{\alpha i}^*$ which implies $f_{\alpha i}^* \leq \frac{1}{\phi_1+\phi_2} T_i^2 f_\alpha$. If the two-step neighborhood average, $T_i^2 f_\alpha$, is nonnegative, $f_{\alpha,i}^* \leq 0$ and $f_i^* \leq \alpha$, $f_i^* \leq \alpha$, i.e., arbitrarily poor local optima of class M_1 cannot exist.
- If $\phi_1 \leq 0$ *and* $\phi_1 + \phi_2 > 0$ it is easily shown that $f_{\alpha i}^* \geq \frac{1}{\phi_1+\phi_2} T_i^2 f_\alpha$. If the two-step neighborhood average, $T_i^2 f_\alpha$, is nonnegative, then $f_{\alpha,i}^* \geq 0$ and $f_i^* \geq \alpha$. A landscape, with respect to the set M_1, is *not favorable* for direct search methods.

We now summarize the above results for all one-step local minima, the x_i^*. Four cases exist:

(1) If $\phi_1 \geq 0$, $\phi_1 + \phi_2 > 0$, and $T_i^2 f_\alpha \leq 0$ for *all* $x_i^* \in M_1 \cup M_2$, then $f_i^* \leq \alpha$, i.e., no arbitrarily poor one-step local minima can exist.

(2) If $\phi_1 \geq 0$, $\phi_1 + \phi_2 < 0$, and $T_i^2 f_\alpha \leq 0$ for *all* $x_i^* \in M_1 \cup M_2$, then every one-step local minimum has $f_i^* \equiv \alpha$, i.e., the global minimum is α.
(3) If $\phi_1 \leq 0$, $\phi_1 + \phi_2 < 0$, and $T_i^2 f_\alpha \geq 0$ for *all* $x_i^* \in M_1$, then no arbitrarily poor one-step local minima, $x_i^* \in M_1 \cup M_2$, can exist.
(4) If $\phi_1 \leq 0$, $\phi_1 + \phi_2 > 0$, and $T_i^2 f_\alpha \geq 0$ for *all* $x_i^* \in M_1$, then arbitrarily poor local minima, $x_i^* \in M_1$, will exist (f_i^* *will* exceed α for one or more i) and the landscape is not favorable for a local search.

4. The characteristic landscape equation for AR(p) landscapes

Here we extend the development of the characteristic landscape equation for AR(2) landscapes to AR(p) landscapes.

Proposition 3. *If the time series based on the random walk on the landscape is consistent with* AR(p) *process, then the landscape satisfies the equation*

$$(T^p - \phi_1 T^{p-1} - \phi_2 T^{p-2} - \cdots - \phi_{p-1} T) f_\alpha = \phi_p f_\alpha.$$

Proof. The proof follows immediately by using the recurrence equation for the AR(p) process

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + \cdots + \phi_p z_{t-p} + a_t$$

and applying the technique used in the proof of Proposition 1. □

Proposition 4. *If the landscape satisfies the equation*

$$(T^p - \phi_1 T^{p-1} - \phi_2 T^{p-2} - \cdots - \phi_{p-1} T) f_\alpha = \phi_p f_\alpha$$

then the time series based on a random walk on this landscape is consistent with the AR(p) *process.*

Proof. The proof follows directly from the proof of Proposition 2. □

5. Conclusions and future research directions

The primary contributions of this work are:

(1) developing the characteristic landscape equation for AR(2) and AR(p) landscapes;
(2) proving that a landscape $\mathcal{L}$ satisfies the corresponding characteristic landscape equation if and *only* if a univariate time series of $f_{\alpha i}$ generated from a random walk on $\mathcal{L}$ is consistent with an AR(p) process;
(3) providing some additional landscapes, exclusive of elementary landscapes, which are favorable for local searches.

Included in our immediate plans for future research are:

(1) investigating the properties of the AR(p) landscape when $p > 2$;
(2) developing the characteristic landscape equations for other than AR(p) time series processes like ARMA, ARIMA, MA.

Another possible area for future research would be to determine whether the hierarchical relationship between the model satisfying the Laplacian equation, $T f_\alpha = \phi_1 f_\alpha$, and the model satisfying Eq. (7) could provide a more efficient computational method for computing the values of ϕ_1 and ϕ_2.

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