



A Note on Characterizing the k -OPT Neighborhood Via Group Theory

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Abstract. Group theory can be used to model and synthesize the neighborhood of Traveling Salesman tours reachable through k -OPT exchanges. A primary concept is that a dihedral group action partitions the sets of cut arcs so that k -OPT exchanges of orbital elements are conjugate. Also presented is a method to produce all k -OPT exchanges for a given set of cut arcs.

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In this paper, we consider group theory perspective associated with a metaheuristic applied to permutation selection problems in combinatorial optimization. Specifically, we show how group theory can succinctly represent the move neighborhood reachable by k -OPT move strategies (Lin and Kernighan, 1973) for an n -city Asymmetric Traveling Salesman Problem (n -ATSP).

When solving an n -ATSP using a metaheuristic method such as tabu search or genetic algorithms, one can use many move strategies that transition from one solution tour to another, e.g., swap two cities, reverse subpaths, or change a city's position. In the k -OPT strategy, k arcs, some or all of which may be adjacent, are cut from the current tour. k possibly different arcs are then used to connect the endpoints of the disconnected subpaths to form a new tour. In this paper, some or all of the cut arcs may be present in the new tour.

Given k and n , this paper shows how group theory may be used to find the k -OPT patterns of a dense n -ATSP and the k -OPT exchanges for a specific set of cut arcs. Since the connection topology alone determines permissible tours, other parameters such as distances between city pairs are not considered here.

1. Group Theory Perspective

The solution space of the dense n -ATSP are the $n!$ permutations of the letters $\{1 \dots n\}$. However, we can take advantage of useful algebraic structure by viewing this space as

$C(n)$, the conjugacy class of n -cycles in S_n —the permutation group on n letters (Isaacs, 1994).

Thus, in the 3-ATSP, the tour 3-2-1-3 is represented by the 3-cycles (3,2,1), (2,1,3), and (1,3,2), all of which are equal in S_3 . The solution space is $\{(1, 2, 3), (1, 3, 2)\}$ and in general, $|C(n)| = (n - 1)!$.

1.1. Notation

For $A, B \subseteq S_n$ and $\rho, \sigma \in S_n$:

- $\rho^\sigma \equiv \sigma^{-1}\rho\sigma$, the conjugate image of ρ under σ
- $A^B \equiv \{a^b : a \in A, b \in B\}$ and $\sigma^*A \equiv \{\sigma a : a \in A\}$
- $\langle A \rangle \equiv$ subgroup generated by A
- $A \leq G \equiv A$ is a subgroup of the group G

1.2. Illustrative Strategies

The following suggests how group theory can represent simple move strategies, where $\sigma \in C(n)$ represents the incumbent tour:

- $\sigma^{(i,j)}$ swaps cities i and j . If swapping two cities is the move strategy, then σ^T are the tours reached from σ in one move, where $T = \{\text{all transpositions in } S_n\}$.
- If one-to-one swaps are made between disjoint cities-subsets A and B (cities a_i and b_i swap), the resulting tour is $\sigma \prod_{(a_i, b_i)}$.
- To reverse a subpath in σ , simply conjugate σ by the product of transpositions determined by endpoints on the contracting subpath. For example, to reverse the subpath 3-4-5-6-7-8-9 in $\sigma = (1,2,3,4,5,6,7,8,9,10,11)$, form the product $\rho = (3, 9)(4, 8)(5, 7)$ to obtain σ^ρ .

2. k – OPT Patterns

The first step to solve an n -ATSP via many metaheuristics is to choose an initial incumbent tour. Once done, the move strategy defines the incumbent’s neighborhood—the tours that can be reached from the incumbent in one move. The “best-suited” neighborhood member then becomes the new incumbent and the process repeats.

2.1. Illustration

With k fixed, a k -OPT exchange can be done in many ways depending on which k arcs are cut. For example, if $\sigma = (1, 2, 3, 4, 5, 6)$ and arc $(3,4)$ is cut, then there is only one way to recover a tour: restore the arc to yield σ and σ^{-1} .

On the other hand, if alternating arcs $\{(1, 2), (3, 4), (5, 6)\}$ are cut, then sixteen different tours can be recovered (eight are inverses of the others). Likewise if alternating arcs $\{(2,3), (4,5), (6,1)\}$ are cut. Although both sets of sixteen differ, the cut “pattern” is the same—cut arcs alternate. As it turns out, exchanges for different cut sets from the same pattern are conjugate and thus “the same.”

2.2. Necklaces

Patterns that spring from the k -OPT strategy are found by solving the n -bead necklace problem (Gilbert, 1976): arcs become beads which are colored green (if cut) and white (if kept). The task is to find all colorwise-distinct necklaces with k green beads, where necklaces can be rotated and flipped.

Colored patterns correspond to the orbits of a group action: the group of symmetries of a regular n -gon (i.e., D_n , the dihedral group on n points) acting on the set of n -bead necklaces with k green beads. This requires the incumbent tour to be the n -cycle $(1 \dots n)$, a necklace with consecutively numbered beads. An orbit’s elements are necklaces that are colorwise the same but whose beads carry their numeric identity.

2.2.1. An Orbit’s Exchanges

Since an orbit’s numbered-necklaces are obtained by rotations and flips, so too are the necklaces’ exchanges—these are conjugate. That is, if OPT_i and OPT_j are respectively the sets of k -OPT exchanges of the i ’th and j ’th orbital members, then $OPT_i = OPT_j^{d_{ij}}$ for some $d_{ij} \in D_n$.

Indeed, the orbit’s full set of exchanges is $OPT_i^{D_n}$, where i is arbitrary. Fortunately, redundant computations may be reduced by instead finding OPT_i^T , where T is any transversal on the right cosets of the “setwise stabilizer in D_n of OPT_i .”

While it may tempting to seek such a transversal T that doubles as a subgroup in D_n , this isn’t always possible. For example, when $n = 8$ and $k = 4$, there is an orbit for which no such transversal T is a subgroup.

2.2.2. All Exchanges

If finding each orbit’s exchanges is required, then their union gives the complete set of exchanges on the incumbent tour. If not, then the following approach gives all exchanges while avoiding some redundancies involved in finding each orbit’s exchanges.

Let $\{OPT_i\}_{i=1}^m$ be a collection of exchanges from each of the m orbits, i.e., OPT_i is the set of exchanges for an arbitrary member of the i 'th orbit. If $S = (\bigcup_{i=1 \dots m} OPT_i)$, then the set of all exchanges for the incumbent tour is given by $E = S^{D_n}$.

As before, we can replace conjugation by D_n by that of T (any transversal on the right cosets of the "setwise stabilizer in D_n of S "). However, this may be wasted effort: if S is large and varied enough (as it may be when it joins inter-orbital exchanges) relative to D_n (which has the "small" size $2n$), then the stabilizer will be the trivial subgroup consisting of the identity in S_n , and so T would be D_n .

2.2.3. Changing the Incumbent Tour

Thus far, the incumbent tour has been the n -cycle $\sigma = (1 \dots n)$. If the incumbent is instead $\rho \neq \sigma$, then the set of exchanges for ρ is given by E^x , where x is any element of S_n that satisfies $\rho = \sigma^x$.

2.3. Specific k -OPTs

The steps below join the endpoints of the surviving one-way disconnected subpaths and then take all possible combinations of subpath reversals. This gives all exchanges for the specific cut arcs. Note that a subpath may be an isolated city obtained from adjacent cut arcs.

The stand-alone development below permits an arbitrary incumbent tour. However, if all k -OPT exchanges for several different tours are desired, then the exchanges for $(1 \dots n)$ should be computed. The process would then proceed per 2.2.3 above.

2.3.1. Clarifying Example

Consider the 16-ATSP with incumbent tour given by the 16-cycle $\sigma = (1 \dots 16)$, and the 7-OPT with cut arcs $\{(4, 5), (5, 6), (6, 7), (10, 11), (13, 14), (14, 15), (16, 1)\}$. The surviving subpaths are $\{[1,2,3,4], [7,8,9,10], [11,12,13], [15,16], [5], [6], [14]\}$.

2.3.2. Notation

Consider the general n -ATSP and a specific k -OPT. Let $B = \{b_i\}$ be the k surviving subpaths and h_i be the head of b_i . Let c_i be b_i expressed as a cycle in S_n . For example, $b_2 = [7, 8, 9, 10]$, $h_2 = 7$, and $c_2 = (7,8,9,10)$.

Let p_i be the product of transpositions formed by the contracting endpoints of b_i . For example, $p_1 = (1, 4)(2, 3)$, $p_2 = (7, 10)(8, 9)$, $p_3 = (11, 13)$, and $p_4 = (15, 16)$. These products will be used to reverse subpaths in a tour, as described earlier.

2.3.3. Method and Implementation

- Let z be the k -cycle $(h_1 \dots h_k)$
- Let $G = \langle z, (h_1, h_2) \rangle$
- Let C be the conjugacy class in G of z
- Let $a = \prod c_i$ (point c_i have no effect and so may be omitted)
- Let $H = \langle p_1, \dots, p_k \rangle$

The exchanges we seek are given by $(a^*C)^H$. That is, a^*C is the set obtained by mating endpoints of the one-way surviving subpaths, while conjugation by H gives all combinations of subpath reversals.

The above computations are readily implemented in GAP v3.4.4—Groups, Algorithms, and Programming. This software is available from <http://www-gap.dcs.st-andrews.ac.uk/~gap/> and is a system for computational discrete algebra with emphasis on computational group theory (Schönert, 1995).

3. Summary

We have shown a group theoretic way to build Traveling Salesman tours via k -OPT exchanges, a move strategy used in a variety of metaheuristics. Characterizing move strategies in group theoretic terms may shed light on the strategies' behavior and properties. This is made possible by the algebraic structure of the symmetric group on n letters.

Finally, group theory offers a compact way to represent the mechanics of move strategies, both conceptually and programmatically. Prototyping move strategies in group theory software may offer advantages over coding in familiar procedural languages.

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References

- Fraleigh, J. (1976). *A First Course in Abstract Algebra*. Reading: Addison-Wesley.
- Gilbert, W. (1976). *Modern Algebra with Applications*. New York: John Wiley & Sons.
- Herstein, I. (1964). *Topics in Algebra*. Waltham: Xerox College Publishing.
- Isaacs, I. (1994). *Algebra: A Graduate Course*. Pacific Grove: Brooks/Cole Publishing.
- Lin, S., and B. Kernighan. (1973). "An Effective Heuristic Algorithm for the Traveling Salesman Problem," *Operations Research* 21, 498–516.
- Schönert, Martin, et al. (1995). *GAP—Groups, Algorithms, and Programming*. Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, fifth edition.