



Local Search Structure in the Symmetric Travelling Salesperson Problem under a General Class of Rearrangement Neighborhoods

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Abstract—The symmetric travelling salesperson problem with n cities (1-STSP) possesses no arbitrarily poor local optima for search neighborhoods defined by arbitrary unions of conjugacy classes in the symmetric group on n letters, $S(n)$. © 2000 Elsevier Science Ltd. All rights reserved.

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Colletti and Barnes [1] detailed how the conjugacy class of n cycles in $S(n)$ may be viewed as the set of all solutions, or tours, to the symmetric travelling salesperson problem with one agent (1-STSP). Large 1-STSPs overwhelm exact solution methods, causing recourse to heuristic methods like the tabu search metaheuristic [2] which iteratively chooses a new solution from among those in the neighborhood of a previously obtained incumbent solution. In this paper, we use group theory to reveal a useful property common to a *general class of rearrangement neighborhoods*.

In earlier work, Grover [3] and Codenotti and Margara [4,5] showed that four specific elementary 1-STSP neighborhoods—2-city swap, 2+3-new-change, 3-new-change, and 2-opt—satisfy a simple homogeneous linear difference equation

$$\nabla^2 f + \frac{k}{n} f = 0, \quad (1)$$

where n is the number of cities; $k > 0$ is a constant which depends on the neighborhood; and for tour p , $f(p)$ is the tourlength of p minus the average tourlength of all tours, μ . Finally, $\nabla^2 f(p)$ denotes the average of all $f(q) - f(p)$, where q is a neighbor of p . Grover [3] shows that the

tourlength of any local optimum of any duplicative 1-STSP neighborhood (not only the above four) does not exceed μ , i.e., arbitrarily poor local optima cannot exist for such neighborhoods. A duplicative neighborhood is one whose construction method admits duplicate members; and henceforth, we presume such neighborhoods.

Let $|C|$ denote the cardinality of $C \subseteq S(n)$. For any n -cycle $p \in S(n)$, define $p^C \equiv \{p^c = c^{-1}pc : c \in C\}$ to be the C -rearrangement neighborhood of p . Note, that each element of C contributes a neighbor, and so p^C is a duplicative neighborhood since there may be distinct $x, y \in C$ such that $p^x = p^y$. When C is a conjugacy class of $S(n)$, [1] showed that in the multiple asymmetric TSP (m-ATSP), the summed neighbor tourlengths—denoted $\text{weight}[\text{tourlength}, p^C]$ —is linear in the tourlengths of p and its inverse. Thus, for the m-STSP and $n > 3$, this weight is linear in $\text{tourlength}(p)$

$$\text{weight}[\text{tourlength}, p^C] = (2c_2 + (n-4)c_4 + 2c_6) \text{Sum } D + [c_1 - 2c_2 + 2c_4 + c_5 - 2c_6] \text{tourlength}(p), \quad (2)$$

where $\text{Sum } D$ is the sum of all elements in the arbitrary symmetric zero-diagonal n by n distance matrix, and c_i is described in [1]. The neighborhood f -weight of p , denoted $\text{weight}[f, p^C]$, is the summed f values of p 's neighbors.

A conjugacy class of $S(n)$ consists of all permutation having a given cycle structure. Thus, the set of all two-cycles in $S(4)$ is the conjugacy class $C_{(x,x)} \equiv \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$, while the conjugacy class of three-cycles is $C_{(x,x,x)} \equiv \{(2, 3, 4), (2, 4, 3), (1, 2, 3), (1, 2, 4), (1, 3, 2), (1, 3, 4), (1, 4, 2), (1, 4, 3)\}$. If p is a derangement in $S(n)$, i.e., a permutation that moves all n letters, then the neighborhood of p consisting of all possible two-letter swaps on p is given by $p^{C_{(x,x)}}$. Likewise, all possible three-letter swaps are given by $p^{C_{(x,x,x)}}$.

However, the set of all four-letter swaps is given by p^C , where $C = C_{(x,x,x,x)} \cup C_{(x,x)(x,x)}$. In general, p^C are all k -letter rearrangements on derangement p , where C is the union of all conjugacy classes whose cycle structures move k letters. Again, p^C may contain duplicates since each element in C creates a neighbor. Until otherwise stated, C henceforth denotes a single conjugacy class of $S(n)$.

When applied to the n -city 1-STSP, equation (2) may be used to prove any C -rearrangement neighborhood satisfies equation (1), and so has no arbitrarily poor local optima. To see this, simply manipulate equation (1) to obtain

$$k = n - \frac{n}{\nu} \frac{\text{weight}[f, p^C]}{f(p)} > 0, \quad (3)$$

where $\nu = |p^C| = |C|$, a constant $\forall p \in S(n)$. Because equation (1) presumes k is constant, then so is

$$\rho = \frac{\text{weight}[f, p^C]}{f(p)} < \nu. \quad (4)$$

Since this argument can be reversed, it follows that a constant $\rho < \nu$ implies the 1-STSP C -rearrangement neighborhood satisfies equation (1).

To show that any C -rearrangement neighborhood respects equation (1), we need only show its ρ value satisfies equation (4). First, it is well known that $\text{SUM } D = (n-1)\mu$, and so

$$\begin{aligned} \text{weight}[f, p^C] &= \text{weight}[\text{tourlength}, p^C] - |C|\mu = [2c_2 + (n-4)c_4 + 2c_6] (n-1)\mu \\ &\quad + [c_1 - 2c_2 + 2c_4 + c_5 - 2c_6] \text{tourlength}(p) - |C|\mu, \end{aligned} \quad (5)$$

where as stated earlier

$$f(p) = \text{tourlength}(p) - \mu. \quad (6)$$

For the reader's convenience, the arc transformation table from [1] is given in Table 1.

Table 1. Arc transformation table.

Transformed arc	τ	$[x, \neq x^p]$	$[\neq x, x^p]$	$[\notin \{x, x^p\} \notin \{x, x^p\}]$	$[x^p, x]$	$[x^p, \neq x]$	$[\neq x^p, x]$
# C -elements	c_1	c_2	$c_3 (= c_2)$	c_4	c_5	c_6	$c_7 (= c_6)$
Arc variates	1	$n - 2$	$n - 2$	$P(n - 2, 2)$	1	$n - 2$	$n - 2$

Table 1 presents the seven transformation of p -arc $\tau = [x, x^p]$ under conjugation by C . c_k elements in C transform τ into a specific column header arc α , and the bottom row gives the number of variants of α . For example, if $\tau = [1, 2]$, then c_2 elements in C change τ into $\alpha = [1, 3]$, one of the $n - 2$ variants $\{[1, 3], [1, 4], \dots, [1, n]\}$. $P(n, m)$ denotes the number of permutations on n choose m letters.

Thus, $|C|$ is the summed pairwise products of corresponding elements from the second and third rows of Table 1

$$\begin{aligned} |C| &= 1 * c_1 + (n - 2) * c_2 + (n - 2) * c_3 + P(n - 2, 2) * c_4 + 1 * c_5 + (n - 2) * c_6 + (n - 2) * c_7 \\ &= c_1 + 2(n - 2)c_2 + (n - 2)(n - 3)c_4 + c_5 + 2(n - 2)c_6. \end{aligned} \quad (7)$$

Substituting equations (5)–(7) into equation (4) and simplifying yields the constant

$$\rho = c_1 - 2c_2 + 2c_4 + c_5 - 2c_6 < |C|. \quad (8)$$

Thus, any 1-STSP C -rearrangement neighborhood satisfies equation (1) with an associated k coefficient

$$k = n \left[1 + \frac{2(c_2 - c_4 + c_6) - c_1 - c_5}{|C|} \right] > 0. \quad (9)$$

In turn, this neighborhood has no arbitrarily poor local optima. Now, if C is the union of arbitrary conjugacy classes $\{C_i\}_{i \in I}$ in $S(n)$, then

$$\frac{\text{weight}[f, p^C]}{f(p)} = \sum_{i \in I} \frac{\text{weight}[f, p^{C_i}]}{f(p)}. \quad (10)$$

Since each summand in equation (10) is constant and less than its $|C_i|$, their sum must be constant and less than $|C|$, the sum of all the $|C_i|$. Thus, this C -rearrangement neighborhood satisfies equation (1) with k coefficient

$$k = n - n \frac{\text{weight}[f, p^C]/f(p)}{\sum_{i \in I} |C_i|} = n \left[1 - \frac{\sum_{i \in I} (\text{weight}[f, p^{C_i}]/f(p))}{\sum_{i \in I} |C_i|} \right]. \quad (11)$$

This result generalizes upon and proves the (specific) conjecture of [4] that says the 1-STSP $\{2 + \dots + m\}$ -letter rearrangement move satisfies equation (1).

In closing, results found here are subsumed by more general m -STSP results presented in [6]. For example, equation (2) holds for the m -STSP, i.e., p need not be an n -cycle (a solution to the 1-TSP). All other equations hold for any deranged m -STSP, i.e., p is any derangement (an n -cycle is a special type of derangement). The authors focused upon the more familiar 1-STSP because once understood, this paper's application to the m -STSP becomes clear.

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