

Note

## A note on distance matrices yielding elementary landscapes for the TSP

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Received 22 August 2001

Symmetric and antisymmetric distance matrices in the single agent traveling salesman problem (TSP) are not the only distance matrices to generate elementary landscapes for “swap” and “2-opt” neighborhoods.

**KEY WORDS:** asymmetric TSP, elementary landscape, neighborhood search

In a TSP with  $n$  cities, there are  $(n - 1)!$  possible tours and a tour’s cost is determined from the  $n \times n$  distance matrix  $\mathbf{D}$  by

$$f_{\mathbf{D}}(\pi) = \sum_{i=1}^n D_{i,\pi(i)}.$$

The *landscape* determined by a neighborhood  $N$  and distance matrix  $\mathbf{D}$  is the pair  $(N, \mathbf{f}_{\mathbf{D}})$ , where  $\mathbf{f}_{\mathbf{D}}$  denotes the vector of tour costs.

Let  $L$  be the  $(n-1)! \times (n-1)!$  Laplacian determined by  $N$  and let  $\tilde{\mathbf{f}}_{\mathbf{D}} = \mathbf{f}_{\mathbf{D}} - \boldsymbol{\mu}$  where  $\boldsymbol{\mu}$  is a vector containing the mean value of  $\mathbf{f}_{\mathbf{D}}$  in each cell. A landscape is *elementary* if  $\tilde{\mathbf{f}}$  is an eigenvector of  $L$ .

Limiting  $N$  to be either a swap or 2-opt neighborhood, Stadler [1] states that if  $\mathbf{D}$  is a symmetric or antisymmetric distance matrix then  $(N, \mathbf{f}_{\mathbf{D}})$  is elementary. The symmetric case is a special case of the more general results presented in Colletti and Barnes [2].

Stadler [1] further claims the converse – if the landscape  $(N, \mathbf{f}_{\mathbf{D}})$  is elementary for swap or 2-opt neighborhoods, then the distance matrix  $\mathbf{D}$  *must* be either symmetric or antisymmetric. As shown below, this latter claim is incorrect.

Define a *deformation pair* to be  $\mathbf{q}, \mathbf{r} \in \mathbb{R}^n$  such that

$$\sum_{i=1}^n q_i + r_i = 0. \quad (1)$$

(Since setting  $r_n$  to be the negative of the sum of *any* choice of  $q_1, \dots, q_n, r_1, \dots, r_{n-1}$  yields a  $(\mathbf{q}, \mathbf{r})$  pair satisfying (1), infinitely many deformation pairs exist.)

**Proposition 1.** Let  $(\mathbf{q}, \mathbf{r})$  be a deformation pair and define

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ q_1 & q_2 & \cdots & q_n \\ \vdots & \vdots & \ddots & \vdots \\ q_1 & q_2 & \cdots & q_n \end{pmatrix} \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} r_1 & r_1 & \cdots & r_1 \\ r_2 & r_2 & \cdots & r_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_n & r_n & \cdots & r_n \end{pmatrix}.$$

If  $\mathbf{D}$  yields an elementary landscape under neighborhood  $N$ , then so does  $\mathbf{D} + \mathbf{Q} + \mathbf{R}$ .

*Proof.* If  $\mathbf{Q}, \mathbf{R}$  and  $\mathbf{D}$  are as defined in the proposition, we only need to show that  $\tilde{\mathbf{f}}_{\mathbf{D}} \equiv \tilde{\mathbf{f}}_{\mathbf{D}+\mathbf{Q}+\mathbf{R}}$ . Since any  $\mathbf{Q} + \mathbf{R}$  satisfying (1) generates a constant TSP, i.e.,  $\tilde{\mathbf{f}}_{\mathbf{Q}+\mathbf{R}} \equiv \mathbf{0}$  (see [3]),  $\tilde{\mathbf{f}}_{\mathbf{D}} \equiv \tilde{\mathbf{f}}_{\mathbf{D}+\mathbf{Q}+\mathbf{R}}$ .  $\square$

**Theorem 2.** Distance matrices yielding elementary landscapes exist which are neither symmetric nor antisymmetric.

*Proof.* Let  $\mathbf{S}$  be any *symmetric* matrix with  $S_{1,n-1} \neq -1/2$ . Setting  $r_1 = 1, q_n = -1, r_i = 0$  for  $i > 1$  and  $q_i = 0$  for  $i < n$  yields a deformation pair, and asymmetric

$$\mathbf{R} + \mathbf{Q} = \begin{pmatrix} 1 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 \end{pmatrix}.$$

$\mathbf{T} = \mathbf{S} + \mathbf{R} + \mathbf{Q}$  is an asymmetric matrix yielding an elementary landscape. Showing that  $\mathbf{T}$  is not antisymmetric completes the proof.

If  $\mathbf{T}$  is antisymmetric,  $T_{1,n-1} = -T_{n-1,1}$ , which implies

$$S_{1,n-1} + 1 = -S_{n-1,1} = -S_{1,n-1}$$

so that

$$2S_{1,n-1} + 1 = 0,$$

contradicting the assumption that  $S_{1,n-1} \neq -1/2$ . Therefore,  $\mathbf{T}$  is a distance matrix yielding an elementary landscape which is neither symmetric nor antisymmetric.  $\square$

This research was supported by a grant from the Air Force Office of Scientific Research. In a forthcoming paper we use the above construction as the basis of a computationally efficient method of recognizing distance matrices which yield an elementary landscape.

## **References**

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