Chapter xx

Using Group Theory to Construct and Characterize Metaheuristic Search Neighborhoods

Bruce W. Colletti and J. Wesley Barnes
Graduate Program in Operations Research and Industrial Engineering
The University of Texas at Austin, Austin TX 78712

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Abstract: Over the past four years, research members of a Consortium composed of The University of Texas at Austin, the Air Force Institute of Technology and the US Air Force Air Mobility Command have been remarkably successful in developing the theoretical foundations of group theoretic tabu search (GTTS) and applying it to a set of complex military logistics problems.[Ψ] This paper briefly recounts some of those accomplishments and describes the underlying mathematical framework that supported them.

The symmetric group on n letters, S_n, can be used to build and solve equations whose coefficients and variables are permutations, i.e., elements of S_n. These equations can efficiently and clearly describe metaheuristic search neighborhoods for combinatorial optimization problems (COPs) whose solutions can be modeled as partitions, orderings, or both partitions and orderings (P|O problems). Following the introduction which provides an overview of the paper's content, the second section describes neighborhoods that preserve a given cyclic solution structure and recounts several examples of how group theory has been used both to provide new and useful insights into the character of such search neighborhoods. The third section describes neighborhoods that embrace multiple cyclic solution structures. Section 4

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considers a hybrid move neighborhood that borrows ideas from Sections 2 and 3. The fifth section overviews some highly successful applications of the GTTS approach for the solution of complex COPs. The sixth and final section contains concluding remarks and offers suggestions for future work.

The two goals of this paper are: (1) to illustrate both that popular metaheuristic move neighborhoods can be cast into a group theoretic context, and that new powerful neighborhoods can be easily "hand tailored" using group theory; and (2) to motivate other researchers to consider how the unifying mathematical framework of group theory could be further exploited to gain powerful new insights and understandings of direct search move neighborhoods.

1. INTRODUCTION

Group theory, the "algebra of permutations", can powerfully enhance the study, understanding and application of metaheuristic search neighborhoods. This statement is abundantly illustrated in the documents referenced in Table 1. In the following sections, each of these papers will be summarized in appropriate detail.

Table 1 Papers Associated with Group Theory & Metaheuristic Neighborhoods

Colletti and Barnes (1999)	Group Theory & Metaheuristic Neighborhoods
Colletti, Barnes and Dokov (1999)	Characterizing the k-OPT Neighborhood Via Group Theory
Colletti and Barnes (2000)	Linearity in the Traveling Salesman Problem
Colletti and Barnes (2001)	Local Search Structure in the Symmetric TSP
Colletti, Barnes and Neuway (2000)	Group theory & Transition Matrices of Metaheuristic Neighborhoods
Barnes, Wiley, Moore and Ryer (2002)	Solving the Aerial Fleet Refueling Problem using GTTS
Crino, Moore, Barnes and Nanry (2002)	Solving The Military Theater Distribution Problem Using GTTS

While the final two papers listed in Table 1 present the use of GTTS in the efficient and effective solution of much more complex COPs occurring in compelling and important *real-world* scenarios, for simplicity and clarity of explanation, we will initially focus our consideration on the special case of the classical m-TSP where the agents do not share a common base or depot city (Gilmore, Lawler, Shmoys 1985). Rather, each agent is based at one of the cities in the subtour, or cycle, assigned to that agent. Let us stipulate an associated arbitrary zero-diagonal real distance matrix and denote this problem as the "cyclic" n-city, m-agent traveling salesperson problem (m-CTSP). (When m = 1, and presuming the depot city is one of the n cities, the m-CTSP becomes identical to the classical 1-TSP.) Some agents may be idle but no more than m cycles are allowed. In this paper, we will use this problem as our illustrative foundation. Colletti and Barnes (1999) show how the m-CTSP results may easily be adapted for use with more general P|O problems such as the classical m-TSP where all agents have a common base.

Although group theory is the foundation of several exact methods of integer and mixed-integer programming (Glover 1969; Gomory 1958, 1965, 1969; Salkin 1975; White 1966; Wolsey 1969), it has found limited use in heuristic and metaheuristic methods. In classical integer programming approaches, the parent group is abelian (commutative). The flexibility of that parent group's many associated factor groups provided a readily available path for the construction of such things as Gomory cuts. However, the parent group for heuristics applied to P|O problems, the symmetric group on n letters, S_n, is nonabelian possessing a single unprofitably small factor group. This lack of factor groups associated with S_n forces markedly different group theoretic approaches from those employed for exact techniques. (For completeness, an appendix briefly sketches the group theoretic concepts essential to the understanding of the body of the paper.)

Section 2 provides a general description of the class of *conjugative rearrangement* neighborhoods that preserve the *cycle structure* of the incumbent solution, i.e., all candidate neighbor solutions must possess the same number of cycles and each cycle must possess the same number of cities as in the incumbent solution. This description is followed by an overview of Colletti and Barnes (2000,2001) and Colletti, Barnes and Neuway (2000) to illustrate how group theory can be used to provide new and useful insights into the character of metaheuristic neighborhoods. In the remainder of Section 2, three other specific types of rearrangement neighborhoods are presented in their equivalent group theoretic form.

Section 3 considers neighborhoods that do not preserve the cycle structure of the incumbent solution and presents a general class of group theoretic search neighborhoods achievable through the application of splitting and welding *templates*. The k-Or neighborhood is given as a specific example of this class and two other representative move neighborhood types are presented in their equivalent group theoretic form.

Section 4 uses group theory to present a hybrid neighborhood that uses ideas from Sections 2 and 3. This hybrid neighborhood reveals a very compact representation of a k-arc exchange neighborhood for arbitrary values of k.

Section 5 presents overviews of Barnes, Wiley, Moore and Ryer (2002) and Crino, Moore, Barnes and Nanry (2002) where the problems attacked, the neighborhoods developed and used, and the relative effectiveness and efficiencies of the solution methodologies are discussed. The sixth and final section provides concluding remarks and a glimpse of future research goals.

2. NEIGHBORHOODS THAT PRESERVE CYCLE STRUCTURE

Group theory may be directly applied to P|O problems by using S_n, whose elements make up all possible partitions and orderings of the n objects composing the solution representation. For example, one group element, or *permutation,* of S_{11} is the partition of the 11 objects onto four subsets, $p = (2,3,7)(1,6,5,4)(9,8,11)(10) \equiv (2,3,7)(1,6,5,4)(9,8,11)$, where, by convention, unit cycles are not explicitly written. p has four subtours or *cycles* in which each letter is mapped into its successor.

- Swap cities **2** & **4** in p = (1,**2**,3,**4**,5,6) to yield $q = p^m = p^{(2,4)} = (1,\mathbf{4},3,\mathbf{2},5,6)$
- Cyclically rearrange **2**,**4**,& **6** in p = (1,**2**,3,**4**,5,**6**) to yield $q = p^m = p^{(2,4,6)} = (1,\mathbf{4},3,\mathbf{6},5,\mathbf{2})$
- Similarly, $q = p^m = p^{(2,6,4)} = (1,\mathbf{6},3,\mathbf{2},5,\mathbf{4})$

Figure 1. Three Examples of Rearrangement Moves

Letter rearrangement moves are often used to build TSP neighborhoods. For example, the 2-letter swap neighborhood of tour p is composed of all tours obtained by swapping all letter pairs in p. Such moves are described via conjugation, i.e., if p, q $\in$ group G, then $p^q \equiv q^{-1}pq$ is the *conjugate of p by q*. Conjugation partitions a group into mutually exclusive and exhaustive *conjugacy classes*. For S_n, a conjugacy class consists of all permutations that share a common cycle structure. Figure 1 gives three examples of simple rearrangement moves for a six city TSP.

When G = S_n, p and p^q have the same cycle structure. In addition to computing $p^q = q^{-1}pq$, p^q may also be found by replacing each letter in p by its image under q. Hence conjugation corresponds to letter rearrangement moves. For example, $[(1,2)(3,4)]^{(3,7)(6,9,1)(4,5)} = (6,2)(7,5)$.

2.1 Letter Rearrangement Moves

The neighborhood of all possible k-letter rearrangements on derangement $p \in S_n$ is given by p^C, where C is the union of all conjugacy classes in S_n whose cycle structures move k letters (any permutation in S_n that moves all n letters is called a *derangement*). For example, all letter pair swaps correspond to C $\equiv$ the conjugacy class of all transpositions. All 6-letter rearrangements correspond to C $\equiv$ the union of conjugacy classes whose cycle structures are (x,x,x,x,x,x), (x,x)(x,x,x,x) and (x,x,x)(x,x,x).

We now recount the results of some recent research associated with letter rearrangement moves (Colletti and Barnes 2000,2001; Colletti, Barnes and Neuway 2000) to provide examples that substantiate the relevance of applying group theory to the study of metaheuristic neighborhoods.

Colletti and Barnes (2000) study the class of general letter rearrangement neighborhoods for the asymmetric TSP under conjugation by a union of conjugacy classes. They use S_n to show that the incumbent tour's neighborhood *weight* (the neighbors' summed tourlengths) is linear in the tourlengths of the incumbent and its inverse (it is important to note that any duplicate neighbors are retained in these computations). If one is concerned about the symmetric TSP, this result confirms that the summed tourlengths of neighborhood solutions is a linear function of the tourlength of the incumbent tour. The coefficients of the tourlengths of the incumbent and its inverse are only dependent upon the elements of C and value of n. The fact that the linear coefficients are not dependent upon the particular distance matrix at hand is enticing. There may very well be deeper knowledge present in this observation that will yield even more powerful general structure between search neighborhoods and metaheuristic methods. Many search strategies are based on the presumption that an incumbent tour’s length will directly reflect its neighborhood weight, i.e., a good tour will be found in a neighborhood of good tours. The above relationship brings to light a counterintuitive fact. Under some letter rearrangement neighborhood types, the neighborhood’s weight is inversely related (has a negative slope) to the incumbent tourlength, i.e., a very inferior solution (with a relatively large tourlength) could have a small neighborhood weight, which implies the existence of short tours in the neighborhood. Interestingly, the swap neighborhood possesses the most positive slope of all such rearrangement neighborhoods. This may explain why so much success has been seen in the application of the swap neighborhood in the literature. The ramifications of this linearity property remain largely unexplored. (One associated result, as described later, is that this property has led to the confirmation of an extended conjecture regarding the quality of local optima in rearrangement neighborhoods (Colletti and Barnes, 2001)). It is now known that for the general asymmetric m-TSP over derangements, linear coefficients are unaffected by the cycle structure of the solution space, a somewhat surprising result. When solutions are not derangements, the linearity is "setwise linear" over solutions that move common letters (regardless of sequencing and cycle structure).

There are other observations. First, with respect to the symmetric m-TSP over derangements, when two rearrangement move methods yield very different neighborhoods, each may still "reach" the global optimum at the same rate. That is, if the methods' linear equations share a common slope, then both have the same rate of change with respect to incumbent tourlength. If so, the larger neighborhood method may offer little advantage.

The second observation is that it is highly likely that other fundamental properties of rearrangement move methods remain to be found using group theory. For example, does the linearity property persist when the rearrangement move is defined in terms of a conjugacy class of a subgroup of S_n instead of an entire conjugacy class of S_n? If so, what is the advantage of using such a smaller conjugacy class? This line of questioning has already led to the study of neighborhoods defined as orbits of group actions, some of whose results appear herein.

A final observation is that the weight of a rearrangement neighborhood can now be swiftly computed (directly) without having to add neighbor tourlengths. In turn, this suggests that group theory may reveal a move neighborhood's macroscopic properties free from the piecemeal evaluation of individual neighbors.

Codenotti and Margara (1992) and Grover (1992) reveal four specific simple symmetric 1-TSP neighborhoods that satisfy Grover's homogeneous linear difference wave equation and so have no arbitrarily poor local optima, i.e., no local optimum will exceed the average tourlength of all feasible solutions.

Colletti and Barnes (2001) use the perspective of group theory to extend these results to a general theory that embraces the above four symmetric 1-TSP neighborhoods as special cases. Indeed, Colletti and Barnes (2001) use group theory to prove that any conjugative letter rearrangement neighborhood for the symmetric 1-TSP has the property of no arbitrarily poor local optima. Colletti (1999) generalizes the above 1-TSP results to the symmetric m-TSP whose solution space is any conjugacy class of derangements (n-cycles are one type of derangement).

This property is as surprising as it is useful: surprising because it simply depends upon the move definition – distance matrix and feasible solution space properties have no influence. The property *always exists* for symmetric m-TSP rearrangement neighborhoods over derangements, an unintuitive result. Instead, intuition wrongly suggests that there may well be some extremely lengthy incumbent whose neighbors' tourlengths are well above average. Thus, if a move strategy ends up with an incumbent whose tourlength is longer than the average, then an improving tour will *always* be found among the 2-city swap neighbors. Finally, as noted before, this property leads to questions about what other fundamental "invariant" properties are possessed by rearrangement neighborhoods and how they can be revealed by group theory.

Without the unifying overview structure provided by group theory, this same result could only be provided by deriving the result *one neighborhood at a time*. Thus, in Colletti and Barnes (2000,2001), the use of group theory reveals two insights into metaheuristic neighborhoods that would be difficult, if not impossible, to obtain in other ways.

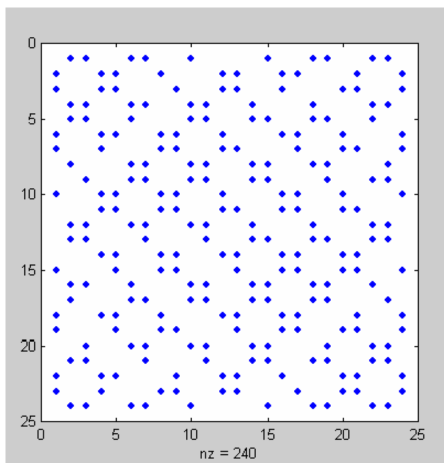

Figure 2. The "Raw" Swap Neighborhood For 5 City 1-TSP

Colletti, Barnes and Neuway (2000) study rearrangement neighborhood transition matrices. For example, consider swap moves (all possible rearrangements of two cities) on a 5 city 1-TSP. In this case solutions are made up of the conjugacy class of 5-cycles, i.e., single agent tours, where the agent visits all five cities (and we assume the agent is based at one of the cities). In this case there are 4! = 24 tours indexed in dictionary order as:

(1,2,3,4,5),(1,2,3,5,4),(1,2,4,3,5),(1,2,4,5,3),(1,2,5,3,4),(1,2,5,4,3),(1,3,2,4,5),(1,3,2,5,4),
(1,3,4,2,5),(1,3,4,5,2),(1,3,5,2,4),(1,3,5,4,2),(1,4,2,3,5),(1,4,2,5,3),(1,4,3,2,5),(1,4,3,5,2),

(1,4,5,2,3),(1,4,5,3,2),(1,5,2,3,4),(1,5,2,4,3),(1,5,3,2,4),(1,5,3,4,2),(1,5,4,2,3),(1,5,4,3,2)

Each of the 24 tours has ten one step neighbors, as pictured in the uninformative schematic transition matrix given in Figure 2. However, by using group theory to find the orbits of a group action of the alternating group, A_n, acting upon all 24 tours, we can use the orbits to partition the tours into two essential classes of the same size. Identically permuting the rows and columns of the transition matrix in accordance with this partition (with some minor reordering) yields the periodic transition matrix of order 2 presented in Figure 3.

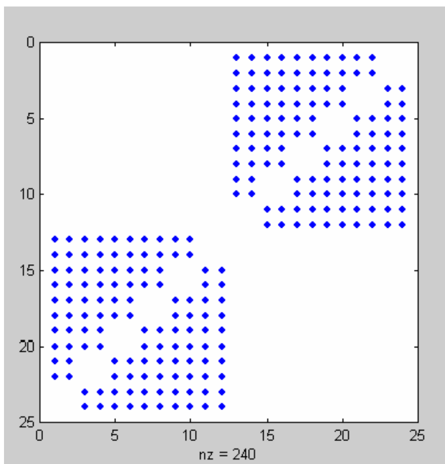

Figure 3. The Transition Matrix ***Revealed****!*

This means that a swap move neighborhood on a 5 city TSP "communicates", i.e., we can get to any other tour starting at an arbitrary tour. However, in one step you can move only from one essential class to the other.

Figure 4 presents the transition matrix, viewed through the perspective of group theory, for the rearrangement neighborhood of order 3, i.e., all possible 3-city rearrangements. This nonintuitive result shows that the 3-city rearrangement neighborhood does not communicate! You can never depart the essential class in which you started. Half the tours are unreachable from any specific tour. If n is an odd number, this surprising structure is present for any n city 1-TSP when the 3-city rearrangement neighborhood is used!

If your starting solution is not in the same essential class with an optimal tour, you can *not* find an optimal solution. However, we may view this structure from a more favorable perspective. If we are aware of this structure, the search for the optimum may be attacked as two independent subproblems with communicating solution neighborhoods of dimensionality $^{(n-1)!}/_2$ by $^{(n-1)!}/_2$ For larger values of n, this is considerably less formidable than the original solution neighborhood with dimension (n-1)! by (n-1)!

Using group theory, Colletti, Barnes and Neuway (2000) prove that for any n-city 1-TSP, if you are using conjugative rearrangement moves with a single conjugacy class defining the move, you either will have a single communicating essential class, or 2 essential classes. If you have 2 classes they will either communicate with period 2 or they will form two noncommunicating classes. They also derive the detailed, specific conditions on the defining conjugacy class and the values of n that lead to each of these structures. (It may be shown that very similar structures are present for the *m-TSP* with n cities.)

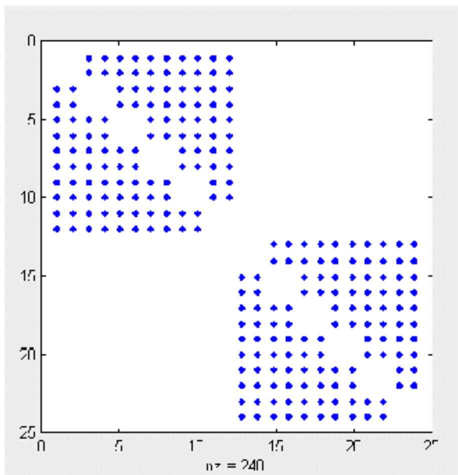

Figure 4. The Transition Matrix for the Order 3 Rearrangement Neighborhood

Using group theory greatly facilitated both the direct discovery of these general structures and proof of their validity for the general n city problem.

2.2 Special Rearrangement Neighborhoods

We now turn our attention to the development of other specific rearrangement moves from the group theory perspective.

2.2.1 Reversal Neighborhoods

Given an m-CTSP tour p, suppose there are specified disjoint subpaths to reverse, e.g., [2,3,4,5,6] and [9,10,11] in p = (1,2,3,4,5,6,7)(8,9,10,11,12) to obtain (1,6,5,4,3,2,7)(8,11,10,9,12). Lin and Kernighan (1973) consider such reversals in their classical heuristic for the symmetric 1-TSP. Group theory allows us to algebraically describe the neighborhood of p, which reverses all combinations of specified subpaths.

Reversing the p-subpath $[a_1,...,a_m]$ requires two steps: first, construct the *contracting endpoints permutation*, $\tau = (a_1,a_m)(a_2,a_{m-1})(a_3,a_{m-2})...(a_i,a_{m-i-1})...$, so named because we start at the extreme ends of the subpath, form a transposition on these letters, and then repeat as we move towards the center of the subpath. Second, compute $q = p^{\tau}$.

Let R be the set of disjoint subpaths to reverse and let $\tau(i)$ be the contracting endpoints permutation of subpath(i). The R-reversal neighborhood of p are the tours that reverse all subpath combinations in p. For example, $p^{\tau(2)\tau(4)}$ simultaneously reverses subpaths 2 and 4, and in general $p^{\prod_{i \in I} \tau(i)}$ simultaneously reverses the indexed disjoint subpaths.

By construction, the disjoint $\tau(i)$'s commute, and each $\tau(i)$ is an involution. Thus if T is the set of all $\tau(i)$'s, then the subgroup generated by T, denoted $\langle T \rangle$, consists of all their possible products. In turn, the reversal neighborhood of p is $p^{\langle T \rangle}$.

2.2.2 Conjugative Elite Paired Solution Neighborhoods

Suppose the feasible solutions of an n-city m-CTSP must satisfy a specific cycle structure, i.e., feasible permutations are elements of a single conjugacy class in S_n. Let p and q be elite feasible tours, where tourlength(q) < tourlength(p). The conjugative equation $p^x = q$ has solution space $X = [\text{Centralizer}(S_n,p)]r$, where r is any specific solution (note that X is a right coset). This fact suggests two neighborhoods which may have superior solutions to both p and q.

The first is $p^{S(mov(\alpha))}$, where α is any element of X. Since $q = p^{\alpha}$, we know the neighborhood has at least one shorter tour than p and perhaps others as well. Note that the α with fewest letters yields the most easily built symmetric group.

For the second neighborhood, q^X, a small $mov(qp^{-1})$ is preferred. In this case, p and q will be very similar and since $p^X = q$ is a shorter tour than p, q^X may include other attractive tours.

2.2.3 Conjugative Path Relinking

Conjugative moves provide a useful way to represent the path relinking metastrategy used to intensify and diversify a tabu search. Glover, Kelly and Laguna (1995) say each intermediate solution in the path from p to q arises from its predecessor in a way that leaves fewest remaining moves to q. As before, we presume p and q share a common cycle structure.

Each solution in S_n to $q = p^x$ represents a "direct move" from p to q. We break down a direct move in order to build the diversification paths of path relinking. That is, if x has multiple disjoint cyclic factors $\{\lambda(i)\}_{i=1...m}$, then one path is $\{p, p^{\lambda(1)}, p^{\lambda(1)\lambda(2)}, ..., p^{\lambda(1)...\lambda(m-1)}, p^x = q\}$. However, the disjoint $\lambda(i)$ commute and so there are at most (m-1)! such paths. The above remarks by Glover, Kelly and Laguna favor the solution x with fewest cyclic factors.

When x is a k-cycle, then express it as a product of transpositions, $x = \Pi\tau_i$ where i takes on values from 1 to k-1. The associated path $\{p, p^{\tau(1)}, p^{\tau(1)\tau(2)}, ..., p^{\tau(1)...\tau(k-2)}, p^x = q\}$ is strictly determined since the $\tau(i)$ don't commute. However, each of the k equivalent representations of x determines a different path.

Thus, either use the x with fewest cyclic factors or the k-cycle x with fewest letters, comparing (m-1)! and k. Although we have assumed the solution space is a conjugacy class, Colletti (1999) describes template-based path relinking when p and q do not share a common cycle structure.

3. TEMPLATE BASED NEIGHBORHOODS

In Section 3.1, we describe a group theoretic process that can transform any permutation in S_n into any other permutation. Specializing the results of Section 3.1, Section 3.2 presents a compact and efficient method to construct neighborhoods that preserve stipulated directed subpaths between cities. A special case of that group theoretic method is then shown to yield a very simple construction of the classical k-Or neighborhood for arbitrary k. Section 3.3 proposes a new neighborhood which seeks to preserve the common properties of two elite solutions.

3.1 Templates

Templates are elements of S_n. Postmultiplying a permutation, p, with the inverse of a *splitting template* τ will partition p into *fragmentary cycles*. The resulting permutation, $q = p\tau^{-1}$, will have more cycles than p. Postmultiplying q by a welding template, ω, will recombine 2 or more of q's cycles to form another permutation, r, with fewer cycles than q. Templates algebraically describe and generalize the classical cross exchange (Taillard, Badeau, Gendreau, Guertin and Potvin 1997) and subtour patching methods (Gilmore, Lawler and Shmoys 1985; Karp 1979; Bartalos, Dudas and Imreh 1995).

For example, consider the 12-cycle, $p=(1,2,3,4,5,6,7,8,9,10,11,12) \in S_{12}$, and the splitting template $\tau = (1,3,6,10,12)$. $q = p\tau^{-1}$ yields $q = (1,2)(3,4,5)(6,7,8,9)(10,11)$. This is the same as removing the arcs {[2,3],[5,6],[9,10],[11,12],[12,1]} from p and then looping each resulting *fragmentary subpath* upon itself. τ is the cycle formed from the p-ordered tails of the subpaths of p that survive the cuts. Alternatively, τ is the cycle formed by the p-ordered heads of arcs deleted in p.

In general, if the $\{\lambda_i\}$ are the disjoint cyclic factors of p, and if τ_i is a splitting template of λ_i, then

$$q = \prod \lambda_i \tau_i^{-1} = \left[\prod \lambda_i\right]\left[\prod \tau_i^{-1}\right] = p \prod \tau_i^{-1} = p\left[\prod \tau_i\right]^{-1}.$$

Thus, if τ is the product of the splitting templates, then $q = p\tau^{-1}$. The above steps use the fact that disjoint permutations commute: the τ_i's are disjoint and when $k > i$, τ_i and λ_k are also disjoint. We also know that for any pair of group elements, $x^{-1}y^{-1} = (yx)^{-1}$.

A *welding template*, ω, is an m-cycle that unites m disjoint cycles according to the template's letter sequence. Template letters come from distinct cycles, which may include 1-cycles. For example, $\{(\underline{1,2,3}),(4,5,6,7),(\underline{\underline{8,9}})\}$ are united by $\omega = (1,4,8)$ to create $(\underline{1,2,3})(4,5,6,7)(\underline{\underline{8,9}})\omega = (\underline{1,2,3},4,5,6,7,\underline{\underline{8,9}})$. If $\omega = (5,2,9)$ then $(\underline{1,2,3})(4,5,6,7)(\underline{\underline{8,9}})\omega = (5,6,7,4,\underline{2,3,1},\underline{\underline{9,8}})$. Note that in the resulting cycle, each factor appears as a subpath with the tail specified in the template.

A *joining template* is a product of disjoint welding templates. Using p, λ_i and τ_i defined earlier, suppose $\{\omega_k\}$ are disjoint welding templates on the fragmentary cycles. If τ is the product of the τ_i's and if ω is the product of the ω_i's, then the m-CTSP tour q created by fragmenting p and then uniting specified fragments is $q = p\tau^{-1}\omega$.

As a special case whose derivation involves splitting and welding templates, consider the classical k-Or neighborhood (Carlton and Barnes 1996). In any m-CTSP tour $p \in S_n$, the k-Or move repositions a k-letter subpath [t,...,h] after letter x to obtain the new tour q. If $\{t,h^p,x^p\}$ are distinct (where y^p denotes the p-image of letter y) and x is not in subpath [t,...,h], then $q = pr$ where r is the 3-cycle (t,h^p,x^p). The k-Or neighborhood is obtained by computing all such q where x satisfies the stated conditions.

For example, consider $p = (1,2,3,4,5,6,7,8)(9,10,11)$ and its subpath [3,4,5,6]. Since $r \in R = \{ (3,7,1), (3,7,2), (3,7,8), (3,7,9), (3,7,10), (3,7,11) \}$, the full 4-Or neighborhood obtained by repositioning the subpath throughout p is pR:

{(1,2,7,8,3,4,5,6)(9,10,11), (1,3,4,5,6,2,7,8)(9,10,11),
(1,2,7,3,4,5,6,8)(9,10,11), (1,2,7,8)(3,4,5,6,9,10,11),
(1,2,7,8)(3,4,5,6,10,11,9), (1,2,7,8)(3,4,5,6,11,9,10) }

3.2 Elite Subpaths Neighborhoods

Suppose that an elite set of directed disjoint subpaths, X, have been identified, i.e., these subpaths are considered so favorable that their integrity should be preserved during the metaheuristic search process. As shown below, group theory makes construction of such a neighborhood straightforward, i.e., the neighborhood is simply a *left coset* of a subgroup, $S(T) \leq S_n$.

Suppose $T \subseteq N = \{1,...,n\}$, and let S(T) be the symmetric group on the letters in T. It is well-known that S(T) can be generated by any transposition in S(T) and some appropriate |T|-cycle, where |T| denotes cardinality of T. We now tie these concepts to templates and the elite subpath neighborhood construction.

If τ is a splitting template on p and if ω is a joining template on the fragmentary cycles of $p\tau^{-1}$, where $mov(\omega) \subseteq mov(\tau)$, then permutation $p\tau^{-1}\omega$ contains the fragmentary subpaths. For example, if $p = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15) \in S_{15}$ and $\tau = (1,4,8,13)$, then $q = p\tau^{-1} = (1,2,3)(4,5,6,7)(8,9,10,11,12)(13,14,15)$ and X = {[1,2,3], [4,5,6,7], [8,9,10,11,12], [13,14,15]}.

If ω is (1,8) or (4,13,1), then $p\tau^{-1}\omega$ is respectively (1,2,3,8,9,10,11,12)(4,5,6,7)(13,14,15) and (4,5,6,7,13,14,15,1,2,3)(8,9,10,11,12). If $\omega = (4,8)(13,1)$, then $p\tau^{-1}\omega = (4,5,6,7,8,9,10,11,12)(13,14,15,1,2,3)$. Thus, any ω whose letters come from the fragmentary tails, T = {1,4,8,13}, is a joining template on the cycles of $p\tau^{-1}$.

In general, given a specific set of disjoint subpaths X, let T be the fragmentary tails of the subpaths in X and let q be the product of the fragmentary cycles. Then S(T) consists of joining templates (on these fragmentary cycles) which preserve the fragmentary subpaths. In turn, all permutations in S_n which preserve the subpaths in X are given by the left coset $qS(T \cup [N - mov(q)])$. The reason for $N - mov(q)$ is that q may not move all letters in N, e.g., when a splitting template isolates cities by cutting adjacent arcs.

For our current example, the neighborhood of p that preserves subpaths in X is the left coset qS(T):

{(1, 2, 3)(4, 5, 6, 7)(8, 9,10,11,12)(13,14,15), (1, 2, 3)(4, 5, 6, 7)(8, 9,10,11,12,13,14,15),
(1, 2, 3)(4, 5, 6, 7, 8, 9,10,11,12)(13,14,15), (1, 2, 3)(4, 5, 6, 7, 8, 9,10,11,12,13,14,15),
(1, 2, 3)(4, 5, 6, 7,13,14,15, 8, 9,10,11,12), (1, 2, 3)(4, 5, 6, 7,13,14,15)(8, 9,10,11,12),
(1, 2, 3, 4, 5, 6, 7)(8, 9,10,11,12)(13,14,15), (1, 2, 3, 4, 5, 6, 7)(8, 9,10,11,12,13,14,15),
(1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12)(13,14,15), (1, 2, 3, 4, 5, 6, 7, 8, 9,10,11,12,13,14,15),

(1, 2, 3, 4, 5, 6, 7,13,14,15, 8, 9,10,11,12), (1, 2, 3, 4, 5, 6, 7,13,14,15)(8, 9,10,11,12),
(1, 2, 3, 8, 9,10,11,12, 4, 5, 6, 7)(13,14,15), (1, 2, 3, 8, 9,10,11,12,13,14,15, 4, 5, 6, 7),
(1, 2, 3, 8, 9,10,11,12)(4, 5, 6, 7)(13,14,15), (1, 2, 3, 8, 9,10,11,12,13,14,15)(4, 5, 6, 7),
(1, 2, 3, 8, 9,10,11,12)(4, 5, 6, 7,13,14,15), (1, 2, 3, 8, 9,10,11,12, 4, 5, 6, 7,13,14,15),
(1, 2, 3,13,14,15, 8, 9,10,11,12, 4, 5, 6, 7), (1, 2, 3,13,14,15, 4, 5, 6, 7)(8, 9,10,11,12),
(1, 2, 3,13,14,15, 8, 9,10,11,12)(4, 5, 6, 7), (1, 2, 3,13,14,15)(4, 5, 6, 7)(8, 9,10,11,12),
(1, 2, 3,13,14,15, 4, 5, 6, 7, 8, 9,10,11,12), (1, 2, 3,13,14,15)(4, 5, 6, 7, 8, 9,10,11,12) }

In summary, Section 4.2 algebraically characterizes the neighborhood that results when we "build all tours preserving given subpaths." The resulting equations can be combined with those representing other metaheuristic concepts and by so doing, reveal insights difficult to obtain via narrative descriptions of algorithms.

3.3 Template-Based Elite Paired Solution Neighborhoods

Suppose that a metaheuristic search has produced elite m-CTSP tours p and q, i.e., p and q are among the most attractive yet found. In the spirit of the path relinking metastrategy (Glover, Kelly and Laguna 1995), intuition suggests p and q may guide us to other attractive tours. One method is to use cosets to mix the inherent properties of the attractive solutions.

Without loss of generality, let tourlength(q) < tourlength(p). It is straightforward to show that $M = mov(qp^{-1})$ are the tails of the noncommon arcs between p and q, a result used to efficiently compute m-TSP tourlength difference (Colletti and Barnes 1999). Thus $qp^{-1} \in S(M)$ and so $q \in S(M)p$, a right coset which may contain other tours shorter than p. Since $|S(M)| = |M|!$, this ancillary neighbor may often merit investigation when M is acceptably small.

As fully detailed in Colletti (1999), group theory may be used to describe many other useful neighborhood construction methods that do not necessarily preserve the cycle structure of the incumbent solution.

4. A HYBRID MOVE

Arc exchange moves combine template and conjugative moves to create 1-TSP neighborhoods. In the k-arc exchange considered here, k arcs are cut from a 1-TSP tour p and replaced by k arcs to recover a new 1-TSP tour q. All subpaths in p that survive the cut will appear unchanged in q. A neighborhood of p on a specific arcs cut set consists of all tours that can be recovered in this way.

Although conceptually straightforward, building such neighborhoods becomes more challenging as k grows modestly, both in programming effort and run time. Using group theory greatly simplifies the description of this neighborhood construction process. The insights provided by group theory, joined with the tools of computational group theory (Schönert 1995), make neighborhood construction straightforward. Although Colletti, Barnes and Dokov (1999) describe this exchange method using a dihedral group action, here we describe it more elegantly using templates and the simpler cyclic group action.

Let the splitting template τ represent the given arc cut set, where $p\tau^{-1}$ is the product of the fragmentary cycles. Although $S(mov(\tau))$ are welding templates which preserve the fragmentary subpaths, not all return a 1-TSP tour. The welding templates that do yield a tour are in $\tau^{S''(mov(\tau))}$, where $S''(mov(\tau))$ is the symmetric group on any $|mov(\tau)| - 1$ letters of $mov(\tau)$. Thus, the k-arc exchange neighborhood of p determined by the splitting template τ is $E_\tau = p\tau^{-1}[\tau^{S''(mov(\tau))}]$.

Now let splitting template λ represent a different k-arc cut. Since λ and τ have the same cycle structure, there is an $r \in S_n$ where $\lambda = \tau^r$. Colletti (1999) shows that the k-arc exchange neighborhood given by λ is

$$E_\lambda = \left[\left(r^{-1}\right)^{\left(p^{-1}\right)} r\right] E_\tau^r .$$

This states E_λ in terms of the *already known* E_τ. We may obtain any E_λ (given E_τ has been obtained) while avoiding construction of $S''(mov(\lambda))$. Furthermore, r can be chosen to have the fewest letters of all elements in the right coset that is the solution space of $\lambda = \tau^x$. These useful but non-intuitive results are possible by viewing these metaheuristic concepts from the group theory viewpoint.

We could build all possible k-arc exchanges by taking the union of all splitting template neighborhoods. However, there is a more effective way that reduces the overlap among neighborhoods. Let $W[\tau] \equiv \tau^{-1}\tau^{S''(mov(\tau))}$ and let T be all ${}_nC_k$ k-cycle splitting templates on p. Let V be a transversal on the orbits of the group action ${}_{<p>}T$, and let R be a transversal on the solution spaces of $\tau^x = v, \forall\, v \in V$. Thus, p's full k-arc exchange neighborhood is

$$N_p = p\left(\bigcup_{r \in R}\left(W^r[\tau]\right)\right)^{<p>}$$

Finally, suppose we seek the full k-arc exchange neighborhood for a new 1-TSP tour q. Since p and q have the same cycle structure, there exists $z \in S_n$ where $p^z = q$ (as before, choose the element (of the right coset solution space) having the fewest letters). Colletti (1999) proves $N_q = N_p^z$. This expresses the full neighborhood for q in terms of the *already computed* full neighborhood for p. Thus, the effort expended to build a full k-arc exchange neighborhood need only be done once during the metaheuristic search. All subsequent neighborhoods can be directly computed via conjugation on the initial tour's neighborhood. Furthermore, if the full neighborhood is built for p = (1,…,n), then this canonical neighborhood can be stored for use in all subsequent searches which build k-arc exchange neighborhoods for the n-city 1-TSP. This reduction in effort holds great promise for improved computational efficiency in evaluating such neighborhoods. Initial *very compelling* indications of the fulfillment of this promise are demonstrated by Jones (2002).

5. EXAMPLE APPLICATIONS OF GTTS

The last two papers referenced in Table 1 were outgrowths of the research proposed by Barnes, Colletti, Fricano, Brigantic, Andrew and Bailey (1999) and are representative of past and current work in employing GTTS to large complex COPs.

Barnes, Wiley, Moore and Ryer (2002) describe a GTTS approach to the Aerial Fleet Refueling Problem (AFRP). A much more detailed treatment is given by Wiley (2001). The AFRP is concerned with the efficient and effective use of a heterogeneous set of tanker (refueling) aircraft, located at various geographical locations, in the required operations associated with the deployment of a diverse fleet of military aircraft (originating in the continental United States) to a foreign location or *theater* of activity. Typically, the “receiving” aircraft must traverse great distances over large bodies of water and/or over other inhospitable environs where no ground based refueling resources exist. Lacking the ability to complete their flights without refueling, each receiving aircraft must be serviced one or more times during their deployment flights by means of in-flight refueling provided by one of the available tanker aircraft. The receiving aircraft, aggregated into receiver groups (RGs) that fly together, have stipulated departure and destination bases and each RG’s arrival time is bounded by a stated desired earliest and latest time. The excellence of a suggested solution to this very challenging decision making problem is measured relative to a rigorously defined hierarchical multicriteria objective function.

Given a deployment scenario, examples of overview questions that require answers are (1) How many tankers are required to meet the air refueling requirements? (2) How quickly can all the receivers be deployed to their final destinations? (3) How far do the tankers and receiver aircraft have to travel? (4) How much fuel is burned by both tankers and by receiver aircraft?

In order to meaningfully answer upper level questions like these, as well as more probing questions relating to efficient planning operations, a great many detailed operational aspects must be addressed.

The following information is assumed to be given: (1) a known set of tankers and their associated original beddown (starting) bases, (2) a known set of RG's, (3) a known set of RG starting and ending bases and tanker beddown bases, (4) a known set of flight characteristics for each aircraft including flight speed, altitude, take-off weight, fuel capacity and fuel-burn rates, and (5) a known set of tanker specific characteristics including fuel-offload capacity and fuel-offload rates.

For a given deployment, the following decisions compose the solution to the AFRP: (1) the waypoints (WPTs), i.e., the physical locations (latitude, longitude and altitude) and start times where each refueling of all RGs will take place (2) the tanker(s) that will serve each WPT, and (3) how much fuel the assigned tanker(s) should deliver to a WPT. We further assume that the decision maker has the authority to (a) stipulate the departure times of all RGs and tankers and (b) to require both tankers and RGs to "orbit" at specified locations to satisfy WPT requirements in terms of timing and location.

As discussed in detail in Barnes, Wiley, Moore and Ryer (2002), the AFRP objective function is multicriteria and hierarchical where the hierarchical ordering of the criteria depends on mission priorities. The solution to the AFRP is composed of a complex set of interrelated decisions involving time, space and amounts of fuel. Usually, the effect of changing any individual decision will "ripple" through the AFRP structure forcing multiple associated changes in the solution.

The AFRP solution is constrained by a large number of limiting factors. The safety of the crews and aircraft associated with the deployment is dominant, i.e., no tanker or receiver aircraft should have to divert from its flight path due to lack of fuel. Many other constraints must also be satisfied. For example, a tanker has limited fuel capacity and its crew has flight duration restrictions which affect the crew-tanker ability to travel long distances and to provide fuel. A tanker requires time to fly between any two locations and time to perform midair refueling. Hence, all tanker WPT assignments must be limited to those where the tanker is physically capable of being present at the specified WPT time.

The AFRP is unique, complicated and extremely difficult to model and solve when viewed in its full practical context. As is discussed in Wiley (2001), the AFRP can be related to other classical combinatorial optimization problems by relaxing specific constraints or by fixing selected decision variables. Perhaps the most closely associated classical problem is a variation of the multi-vehicle, multi-depot Vehicle Routing Problem (VRP).

However, there are several additional considerations present in the AFRP that are not present in VRP. Among these added considerations are that the AFRP "customers" (RGs) are not fixed in space but rather have multiple time dynamic locations. These time and locations and the amount of product to be delivered must all be determined as part of the solution. In addition, all RGs must be supplied with fuel in a timely manner to prevent "untimely" flight diversions. Directly associated with the WPT decisions are the decisions on the takeoff time of each RG and the possibly multiple takeoff times of each tanker.

The primary objective of the research documented in Barnes, Wiley, Moore and Ryer (2002) was to develop methods for producing an "ensemble" of excellent solutions to any instance of the AFRP. To achieve this objective a Group Theoretic Tabu Search (GTTS) approach was created. A solution to the AFRP was represented as an element of S_n and move neighborhoods were also represented by elements of S_n acting under conjugation or multiplication. To address the issue of reusability and portability, the GTTS approach was implemented using the Java programming language. The implementation makes extensive use of Wiley's (2000) Java class library for S_n.

In the S_n solution depiction, each tanker's deployment activity was represented as a single cycle in the associated symmetric group permutation. Letter or "node" types were associated with tankers, WPTs and airbases in such a way that 8 *specifically tailored* dynamic neighborhoods could be used to search the complex solution space. The construction of such dynamic neighborhoods, which were *essential* to the efficient and effective solution of the AFRP, would not have been practical without the group theoretic perspective that was employed.

Using this GTTS-AFRP methodology, a number of deployment scenarios were attacked. The computational results show that the GTTS-AFRP is effective, providing good results with no parameter tuning. The solutions that were obtained were superior to all known benchmark problem solutions and the procedures were sufficiently robust to allow diverse objective functions and constraints to be considered with no additional difficulty. In particular, a practical sized (typical) middle east deployment scenario was attacked. This problem consisted of 99 aircraft within 26 RGs originating at bases within the continental US utilizing 6 tanker origination/beddown bases with 120 available tankers.

The best solution, found in two hours and 16 minutes (on a AMD Athlon 950 Mhz machine with 256 MB RAM), employed 95 tankers flying 326,968 miles and allowed the latest RG to arrive in 55 hours. The tanker assignments in this solution were obtained at the *detailed asset operational* *level* (individual tankers or receivers) and yielded excellent assignments in terms of such metrics as minimizing tanker usage, tanker travel and total deployment time. In general, depending on the desire of the user, the GTTS-AFRP method can provide either a single "best" solution to a particular problem or a set of robust, comparable solutions to a problem instance. The fact that the best previous method available to the US Air Force would require several analysts weeks, or months, to achieve a single feasible solution strongly underscores the power of the GTTS-AFRP. Thus, with this new tool, analysts and decision makers not only can have the flexibility and added insight provided by multiple solutions, they also can make critical decisions in a tremendously shortened planning horizon.

The last paper referenced in Table 1 (Crino, Moore, Barnes and Nanry (2002)) presents a GTTS approach to the Theater Distribution Vehicle Routing and Scheduling Problem (TDVRSP). The TDVRSP is associated with determining superior allocations of required flows of personnel and materiel within a defined geographic area of military operation. A theater distribution system is comprised of facilities, installations, methods and procedures designed to receive, store, maintain, distribute and control the flow of materiel between exogenous inflows to that system and distribution to end user activities and units within the theater. An automated system that can integrate multi-modal transportation assets to improve logistics support at all levels has been characterized as a major priority and immediate need for the US military services.

Crino, Moore, Barnes and Nanry (2002) describes both the conceptual context, based in a flexible group theoretic tabu search (GTTS) framework, and the software implementation of a robust, efficient and effective generalized theater distribution methodology. The concepts and results are presented in much greater detail in Crino (2002). This unprecedented methodology, for the first time, evaluates and determines the routing and scheduling of multi-modal theater transportation assets at the *individual asset operational level* to provide economically efficient time definite delivery of cargo to customers.

Modeling requirements for the TDVRSP extend well beyond those of the typical VRP. Within a typical theater, there are multi-modal vehicles with differing characteristics such as cruising speed, capacity, cruising length, fixed and variable costs, route type (air, ground, water) and service times. Vehicle modeling requirements include the ability to make one or more trips during the planning horizon, the ability to perform direct delivery from outside the theater, and the ability to refuel at customer locations. Scheduling considerations include vehicle availability, service times, load times and unload times. All vehicles operate from a depot or a hub.

The theater distribution network nodes are depots, hubs and customers. Depots, aerial ports of debarkation (APODs) and seaports of debarkation (SPODs), are the supply nodes. The hubs, or support areas (SAs) which occur at corps, division and brigade levels, are transshipment nodes, which receive, store and distribute cargo. Customers are sink nodes that receive cargo. All nodes have time window constraints, fuel storage capacity and maximum on the ground (MOG) constraints. Hubs have cargo storage constraints and customers have cargo demand requirements and time definite delivery requirements.

A TDVRSP has three types of time window constraints: early time definite delivery (ETDD), time definite delivery (TDD) and multiple time windows for non-departure and non-arrival times (MTW). An ETDD stringently defines a customer service starting time but does not constrain vehicle arrival or departure times. A TDD defines when a customer service should be complete but does not constrain service occurrence, or vehicle arrival and departure times. MTWs restrict vehicle arrival and departure at a node but do not stipulate when vehicles are loaded or offloaded.

The TDVRSP has a tiered distribution architecture. The first order tier contains the depots and customers/hubs served by the depots. Middle tiers consist of hubs that service customers/hubs. The last order tier consists of end customers served by a hub. Each tier is a self-contained distribution network and lower ordered tiers are dependent on higher ordered tiers.

Hubs distribute cargo after it is received and processed. Cargo, characterized by the amount delivered and time of delivery, is processed and prepared for delivery to its next customer. Cargo is either loaded directly onto available vehicles or stored for later delivery.

The TDVRSP primary objectives are to minimize unmet customer demand (demand shortfall), late deliveries (TDD shortfall), vehicle fixed costs and vehicle variable costs. Late delivery times are weighted by the amount of demand delivered late. A fixed cost is charged for each vehicle used in the solution and variable vehicle costs are associated with vehicle travel.

Applying GTTS to the TDVRSP requires that solutions be represented as elements of S_n. For a TDVRSP solution, each cycle in the solution's disjoint cyclic structure represents a vehicle trip where the first letter in the cycle represents a vehicle. Subsequent letters represent the customers serviced by that vehicle. A unit cycle represents either a vehicle letter not leaving the depot/hub or a customer letter not serviced.

Since vehicles can make multiple trips and customers can receive multiple services within a time period, a unique letter is allocated for each vehicle trip and each customer service

In addition to yielding an efficient and effective solution methodology for the TDVRSP, the research documented in Crino, Moore, Barnes and Nanry

(2002) also provided *significant* advances in the development of GTTS applied to vehicle routing and scheduling problems. These GTTS advances, described in detail in Crino (2002), included, for the *first* time, the formulation of move neighborhoods defined using groups to generate orbits. These orbits were used as a means to efficiently search the solution space. They eliminated cycling, prevented solution reevaluation and avoided entrapment in local optima. This methodology prevents the search from being trapped in a subset of the solution space topology and eliminates the need for a restart mechanism. This technique allowed exhaustive non-repetitive search of each partition and, by placing a utilized orbit on an orbit tabu list, prevented reevaluation.

A unique solution space partition hierarchy was developed using the symmetric group on n- letters. Conjugacy classes, cyclic form structures, orbital planes and orbits were defined that partition the solution space. Solution space partitions were exploited in the diversification and intensification process. In addition, neighborhoods were constructed to intelligently traverse the partitions and enable a potential search of the entire space. Group move neighborhoods steered the search between different orbits. And swap move neighborhoods traversed the search between different orbital planes. Insert and extraction move neighborhoods moved the search to different conjugacy classes and cyclic form structures.

Orbital planes were defined and used as a primary search mechanism of the GTTS. Orbital planes are orbits partitioned by orbits. They provide a more atomic partitioning of the solution space permitting partial or exhaustive search. Using orbital planes is a highly efficient special case of Colletti's (1999) orbital transversal method because the search may re-examine an orbit (orbital plane) without reevaluating solutions within the orbit (orbital plane).

Details on the TDVRSP modeling assumptions and algorithmic solution procedures are discussed in detail in Crino (2002) and Crino, Moore, Barnes and Nanry (2002). Prior to Crino's (2002) research, no benchmark problems existed for the TDVRSP. Crino (2002) used valid experimental design methodology to construct a set of 39 problems that effectively test the robustness of the GTTS-TDVRSP algorithm. The problems are composed of three TDVRSP types: the Air Force multiple trips multiple services (MTMS) without hub, the Joint MTMS without hub, and the Joint MTMS with hub and other defining constraints. The 39 problems were further categorized by problem size (small, medium and large), delivery restriction density (low, medium and high), demand to capacity ratio (low, medium and high).

Results from Crino's benchmark problem 37, one of the *large* Joint MTMS problems with hub and other defining constraints is indicative of the timely, effective and robust results provided by the GTTS-TDVRSP approach for all 39 benchmark problems. The total run time was 110 minutes, where the best solution was found at iteration 709 (62.8 minutes). However, competitive satisfactory solutions, where all customer demands were satisfied, were found much earlier in the search process. The GTTS-TDVRSP method has clearly displayed its ability to solve the multiple objective TDVRSP problem.

The development of an automated solution methodology to the TDVRSP problem has been characterized as a major priority and immediate need for the US military services. Many software programs are available that "perform" theater distribution modeling. However, all, but the technique presented in Crino, Moore, Barnes and Nanry (2002), are simulation models and simulations can provide neither total asset visibility (TAV) nor in-transit visibility (ITV). Therefore, this model is the *first* of its kind to offer this functionality.

In a recent study funded by HQ USAF (HQ USAF/ILA, 2002), a number of recommendations were made to the Air Force on types of models that could support an automated theater distribution management system. The purpose of the system is to "optimize" the entire theater distribution system, from the number of bases and vehicles, down to the vehicle routes and schedules. They concluded that vehicle routing and scheduling was very difficult to optimize, and development of an optimization approach was considered a high-risk methodology. Therefore, they proposed a low risk method using simulation. Unfortunately, they lose the functional requirements of providing TAV and ITV when using simulation. The HQ USAF study validates the importance and magnitude of this research. What was regarded as *too difficult* has been successfully *created, developed and demonstrated* in this TDVRSP research.

6. CONCLUDING REMARKS

Group theory provides a unifying mathematical framework in which to study metaheuristic neighborhoods and search methods applied to P|O problems. This algebra allows us to build and solve equations whose coefficients and variables are permutations and sets of permutations. By manipulating these equations using the comprehensive structure provided by group theory, we can obtain significant understanding of important concepts that are more difficult or impossible to glean using more customary approaches.

Of equal, or greater importance, the understanding and structure provided by the group theoretic perspective can also lead directly to more powerful solution methodology applied to complex practical COPs.

We are continuing our ongoing investigations into the structure and character of several classes of metaheuristic search neighborhoods. We hope that the first steps described in this paper will motivate other researchers to consider how the unified framework of group theory could be further exploited to gain powerful new insights into move neighborhoods and other metaheuristic concepts.

We are also continuing to develop approaches to provide solutions to other complex military logistics problems. As documented in Barnes, McKinzie and Gomes (2002) and in the recently funded Phase II continuation proposal (Barnes, Shirley, Irish, Andrew and Moore 2002) of the efforts proposed by Barnes, Colletti, Fricano, Brigantic, Andrew and Bailey (1999), five additional military logistics problems are being approached using GTTS methodology.

We hope that future researchers will join us in this exciting new area of theoretical and applied work.

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Appendix

The pure mathematics of group theory (Gaglione 1992, Isaacs 1994) has many practical uses. A group is any set and operation which together satisfy the properties of closure, associativity, right-identity and right-inverse (g^{-1} is the inverse of group element g). For example, the set of all n-by-n invertible real matrices is a group: multiplying any two such matrices produces another matrix (*closure*); multiplication is *associative*; right-multiplying a matrix by the *identity* matrix produces the given matrix; and right-multiplying a matrix by its *inverse* produces the identity matrix. Another familiar group is the integers under addition.

Group theory may be directly applied to the m-CTSP by using S_n, whose elements make up all possible partitions and orderings of the n cities. For example, one group element, or *permutation,* of S_{11} is the partition of the 11 cities onto four agents, p = (2,3,7)(1,6,5,4)(9,8,11)(10) ≡ (2,3,7)(1,6,5,4)(9,8,11), where, by convention, unit cycles are not explicitly written. p has four subtours or *cycles* in which each letter is mapped into its successor (denoted $2^p = p(2) = 3$, $3^p = p(3) = 7$, $7^p = p(7) = 2$, $10^p = p(10) = 10$). mov(p) are the letters contained in the disjoint non-unit cycles of p, i.e., for p = (2,3,7)(1,6,5,4)(9,8,11)(10), mov(p) = {1,2,3,4,5,6,7,8,9,11}. If q is also a permutation then the product pq is the composite function produced by applying p and then q, i.e., for letter x, $x^{pq} = (x^p)^q$. For example, if q = (3,7,8,10)(4,9) then $3^{pq} = (3^p)^q = 7^q = 8$, and so under pq, 3 is mapped into 8. Permutation multiplication need not be commutative since $3^{qp} = (3^q)^p = 7^p = 2$, and so pq ≠ qp. Since a permutation represents a 1-1 mapping from the letters {1,...,n} onto itself, multiplication (i.e., function composition) is associative and closed.

The inverse of a permutation simply reverses each cycle, and the identity permutation is that which maps each letter into itself, i.e., is composed of n unit cycles. Thus, the four properties of a group are satisfied and the set of all n! permutations under multiplication is called the symmetric group on n-letters, denoted S_n.

The components of a permutation are cycles and an m-cycle has m letters. A 2-cycle is also called a *transposition* and an n-cycle is a permutation with a single cycle equivalent to a 1-TSP tour. Every permutation is a unique product of *disjoint* cycles, i.e., cycles sharing no common cities. The number of cycles and the cycle sizes define the permutation's *cycle structure* and every cycle is a non-unique product of transpositions. For example, (1,2,3,4) = (2,3,4,1) = (1,2)(1,3)(1,4) while (2,3,4,1) = (2,3)(2,4)(2,1). If p^2 = pp is the identity, then p is an *involution*.

Other important concepts are:

If group H $\subseteq$ group G then H is a *subgroup* of G, denoted H $\leq$ G

If H $\leq$ G and g $\in$ G, then Hg $\equiv$ {hg: h $\in$ H} is a *right coset* of H in G, and gH $\equiv$ {gh: h $\in$ H}is a *left coset* in G

All left (right) cosets of H exclusively and exhaustively partition G

If H $\leq$ G and g $\in$ G, the *Centralizer* of g in H is the set of all elements in H that commute with g, i.e., Cent(H,g) = { h$\in$H: h^{-1}gh = g} = { h$\in$H: gh = hg}

A *transversal* on disjoint sets is a collection of elements, precisely one from each set

The last overview item is the *group action* $_G$T, a concept that uses a group G to partition a set T into mutually exclusive and exhaustive cells called *orbits*. Regarding group elements as "moves" between elements of T, the elements in the partition of x $\in$ T are those reachable from x by any series of moves. If g $\in$ G, then x^g denotes the element in T reached in one step from x via g.

The group action *operator* is the rule that assigns value to x^g, e.g., conjugation if T $\subseteq$ G, the mapping operator if T are letters and G is a permutation group, or similarity products if T and G are n by n matrices. Finally, a group action must satisfy certain properties in order to be valid (Isaacs 1994), i.e., one cannot freely match any group G with any set T. Colletti (1999) uses group actions to escape the chaotic attractors of reactive tabu search (Battiti and Tecchiolli 1994).