Laplace Transforms & Transfer Functions

Laplace Transforms: method for solving differential equations, converts differential equations in time t into algebraic equations in complex variable s

Transfer Functions: another way to represent system dynamics, via the s representation gotten from Laplace transforms, or excitation by e^{st}

Laplace Transforms

Purpose: Converts linear differential equation in t into algebraic equation in s

- Forward transform, $t \rightarrow s$: $f(t) \Rightarrow F(s)$

$$F(s) = \int_{t=0}^{\infty} f(t)\, e^{-st}\, dt = \mathcal{L}\{f(t);\, t \rightarrow s\}$$

Convergence/existence of integral:

$|f(t)| < M e^{-at}$, $a + Re(s) > 0$

so that $e^{-[a + Re(s)]t}$ finite as $t \rightarrow \infty$

- Inverse transform, $s \rightarrow t$: $F(s) \Rightarrow f(t)$

$$f(t) = \frac{1}{2\pi j} \int_{\omega = c - j\infty}^{c + j\infty} F(s)\, e^{st}\, ds = \mathcal{L}^{1}\{F(s);\, s \rightarrow t\}$$

Integral in complex plane, rarely do

Procedure:

- Convert Linear Differential Equations

$$\dot{x} = Ax + f(t)$$

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_2\frac{d^2 x}{dt^2} + a_1\frac{d\,x}{dt} + a_0 x = f(t)$$

  in time t into algebraic equations in complex variable s, via forward Laplace transforms:

  $X(s)=\mathcal{L}\{x(t);\ t \rightarrow s\}$, $F(s)=\mathcal{L}\{f(t);\ t \rightarrow s\}$

- Solve resulting algebraic equations in s, for solution $X = X(s)$

- Convert solution $X(s)$ into time function $x(t)$, via inverse Laplace transform:

$$x(t) = \mathcal{L}^{-1}\{X(s);\ s \rightarrow t\}$$

Example: Solve initial value problem, differential equation with initial condition:

$$\ddot{x} + 2\,\zeta\,\omega_n\,\dot{x} + \omega_n^2\,x = H_0\,u_s(t)$$

$$x(0) = x_0\ ,\quad \dot{x}(0) = x_1$$

- Apply Laplace transform

$\mathcal{L} = \int_{t=0}^{\infty} \{\,\cdot\,\}\, e^{-st}\, dt$ to all terms in equation:

$$\int_{t=0}^{\infty} \ddot{x}(t) e^{-st} dt + \int_{t=0}^{\infty} 2\zeta\omega_n \dot{x}(t) e^{-st} dt + \int_{t=0}^{\infty} \omega_n^2 x(t) e^{-st} dt$$

$$= \int_{t=0}^{\infty} H_0\, u_s(t)\, e^{-st}\, dt$$

- factor constants and apply definition for $u_s(t)$:

$$\int_{t=0}^{\infty} \frac{d^2x}{dt^2}\, e^{-st}\, dt + 2\zeta\omega_n \int_{t=0}^{\infty} \frac{dx}{dt} e^{-st}\, dt + \omega_n^2 \int_{t=0}^{\infty} x\ \ e^{-st}\, dt$$

$$= H_0 \int_{t=0}^{\infty} e^{-st}\, dt$$

- Define $X(s) = \int_{t=0}^{\infty} x(t) \; e^{-st} \, dt$
- Integrate by parts

$$\frac{dx}{dt} \; e^{-st} \Big|_{t=0}^{\infty} + s \int_{t=0}^{\infty} \frac{dx}{dt} \; e^{-st} \, dt + 2\zeta\omega_n \{ x(t) \, e^{-st} \Big|_{t=0}^{\infty}$$

$$+ s \int_{t=0}^{\infty} x(t) \, e^{-st} \, dt \; \} + \omega_n^2 \; X(s) = \frac{H_o}{(-s)} \, e^{-st} \Big|_{t=0}^{\infty}$$

- Rearrange, note $\lim_{t \to \infty} e^{-st} = 0$

$$- \dot{x}(0) + s\,(- x(0) + s\,X(s)\,) + 2\zeta\omega_n \{- x(0) +$$

$$s\,X(s)\,\} + \omega_n^2 \; X(s) = \frac{H_0}{s}$$

  - <u>Note</u>: $\mathcal{L}\{dx/dt\} = - x(0) + s\,X(s)$

$\mathcal{L}\{d^2x/dt^2\} = \mathcal{L}\{d\dot{x}/dt\} = - \dot{x}(0) + s\mathcal{L}\{dx/dt\}$

- Result: algebraic equation in s

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) X(s) = \frac{H_o}{s} + x_1 + (s + 2\zeta\omega_n) x_0$$

Note: $\dot{x}(0) = x_1$, $x(0) = x_0$

- Solve

$$X(s) = \frac{\frac{H_0}{s} + x_1 + (s + 2\,\zeta\,\omega_n)\,x_0}{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2}$$

- Note:
  - o characteristic equation in denominator
  - o terms from initial conditions x_1 , x_0 grouped with excitation term $\frac{H_o}{s}$

- Apply inverse Laplace: $x(t) = \mathcal{L}^{-1}\{X(s);\, s \rightarrow t\}$

$$x(t) = \frac{H_0}{\omega_n^2}\,\mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s\,(s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2)}\right\}$$

$$+ \frac{x_0}{\omega_n^2}\,\mathcal{L}^{-1}\left\{\frac{s\,\omega_n^2}{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2}\right\}$$

$$+ \frac{x_1 + 2\,\zeta\,\omega_n\,x_0}{\omega_n^2}\,\mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2}\right\}$$

○ Use Laplace transform tables for $\mathcal{L}^{-1}$:

$$x(t) = \frac{H_0}{\omega_n^2}\left\{1 - \frac{e^{-\zeta\,\omega_n\,t}\,\sin(\omega_n\sqrt{1-\zeta^2}\,t + \arccos\zeta)}{\sqrt{1-\zeta^2}}\right\}$$

$$+ \frac{x_0}{\omega_n^2}\left\{-\frac{\omega_n^2\;e^{-\zeta\,\omega_n\,t}\,\sin(\omega_n\sqrt{1-\zeta^2}\,t - \arccos\zeta)}{\sqrt{1-\zeta^2}}\right\}$$

$$+ \frac{x_1 + 2\,\zeta\,\omega_n\,x_0}{\omega_n^2}\left\{\frac{\omega_n\;e^{-\zeta\,\omega_n\,t}\,\sin(\omega_n\sqrt{1-\zeta^2}\;t)}{\sqrt{1-\zeta^2}}\right\}$$

Transfer Functions

- Method to represent system dynamics, via *s* representation from Laplace transforms. Transfer functions show flow of signal through a system, from input to output.

- Transfer function *G(s)* is ratio of output *x* to input *f*, in *s*-domain (via Laplace trans.):

$$G(s) = \frac{X(s)}{F(s)}$$

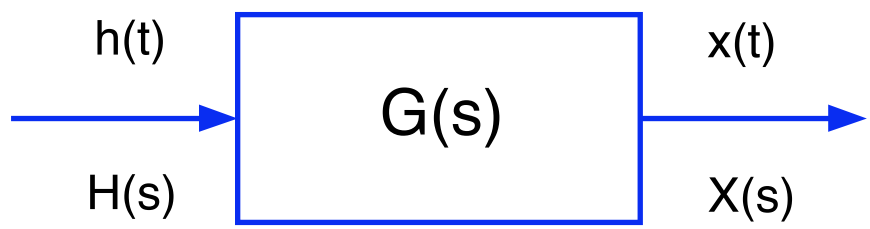
- Method gives system dynamics representation equivalent to

    - Ordinary differential equations
    - State equations

- Interchangeable: Can convert transfer function to differential equations

Transfer Function *G(s)*

- Describes dynamics in operational sense
- Dynamics encoded in *G(s)*
- Ignore initial conditions (I.C. terms are "transient" & decay quickly)
- Transfer function, for input-output operation, deals with steady state terms



- Example: Speed of automobile
  - Output: speed *x(t)* expressed as *X(s)*
  - Output determined by input, system dynamics & initial conditions
  - Input: gas pedal depression *h(t)*
  - Dynamics: fuel system delivery, motor dynamics & torque, transmission, wheels, car translation, mass, wind drag, etc.
  - Initial conditions' (initial speed) influence on current speed diminishes over time, thus ignore

Transfer Function Example: 2nd order system

- Differential equation at steady state (can ignore initial conditions)

$$\ddot{x} + 2\,\zeta\,\omega_n\,\dot{x} + \omega_n^2\,x = h(t)$$

- Apply Laplace transform, define

$$X(s) = \mathcal{L}\{x(t);\, t \to s\}, \quad H(s) = \mathcal{L}\{h(t);\, t \to s\}$$

- Result: algebraic equation

$$(s^2 + 2\zeta\omega_n\, s + \omega_n^2)X(s) = H(s)$$

- Transfer function *G(s)*: ratio of output *X(s)* to input *H(s)*

$$G(s) = \frac{X(s)}{H(s)} = \frac{1}{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2}$$

- Note:
  - o characteristic equation in denominator
  - o denominator roots => eigenvalues
  - o transfer function's eigenvalues called *poles*

Example: First order system

$$\tau\dot{x} + x = f(t)$$

- Apply Laplace transform, define

$X(s) = \mathcal{L}\{x(t);\, t \to s\}, \quad F(s) = \mathcal{L}\{f(t);\, t \to s\}$

$\mathcal{L}\{\tau\,\dot{x};\, t \to s\} + \mathcal{L}\{x(t);\, t \to s\} = \mathcal{L}\{f(t);\, t \to s\}$

- Result: algebraic equation

$s\tau X(s) + X(s) = F(s)$

- Transfer function

$$G(s) = X(s)/F(s) = \frac{1}{\tau s + 1}$$

- Note:
  - characteristic equation in denominator
  - transfer function's eigenvalues = *poles*
  - $p_1 = \lambda_1 = -1/\tau$

Transfer Function from State Equations

- Matrix form

$$\dot{x} = Ax + f(t)$$

- Explicit equation form

$$\dot{x}_k = \sum_{j=1}^{n} a_{jk} x_j + f_k(t)$$

- Define

$X_k(s)=\mathcal{L}\{x_k(t);\, t \to s\}$, $\quad F_k(s)=\mathcal{L}\{f_k(t);\, t \to s\}$

- Apply Laplace transform

$$sX(s) = AX(s) + F(s)$$

or

$$sX_k(s) = \sum_{j=1}^{n} a_{jk} X_j(s) + F_k(s)$$

- Rearrange matrix form:

$$\left[sI - A\right]X(s) = F(s)$$

- Solve, via Cramer's rule:

$$X_k(s) = \frac{\det\left\{\left[sI - A\right]_{k\text{th column replaced by } F(s)}\right\}}{\det\left[sI - A\right]}$$

- Transfer function, define which output $X_k(s)$ and which input $F_j(s)$:

$$G_{kj}(s) = \frac{X_k(s)}{F_j(s)}$$

- Solve, via previous result with all components of $F(s)$ zero, except $F_j(s)$

Example: differential equations from transfer function:

$$G(s) = \frac{X(s)}{F(s)} = \frac{2s+5}{s^4 + 3s^3 + 2s^2 + 2s + 1}$$

- Cross multiply ratios:

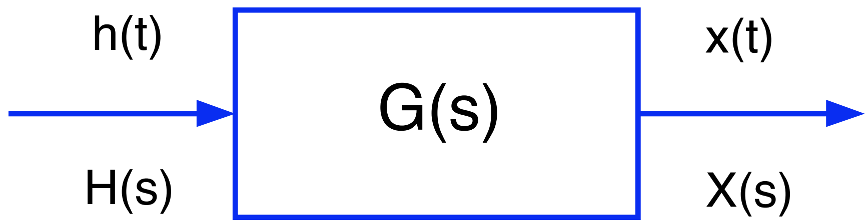
$$\left(s^4 + 3s^3 + 2s^2 + 2s + 1\right)X(s) = \left(2s+5\right)F(s)$$

$$s^4 X + 3s^3 X + 2s^2 X + 2sX + X = 2sF + 5F$$

- Treat: $s \Rightarrow d/dt$

$$\frac{d^4x}{dt^4} + 3\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 2\frac{df}{dt} + 5f(t)$$

Block Diagrams



- Another way to represent system dynamics pictorially

- Weakness: lacks causality information

- Shows signal flow through system

- Transfer function *G(s)* inside block

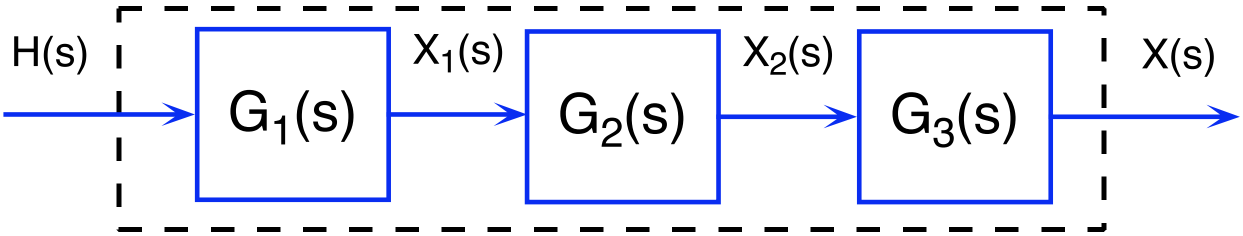
- Output:

$$X(s) = G(s)\ H(s)$$

  transfer function times input *H(s)*

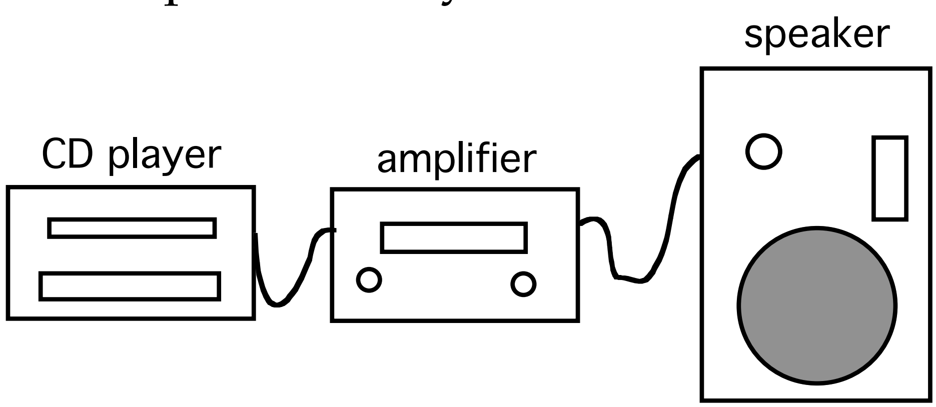
- Can assemble blocks into system model:

- Cascaded Blocks

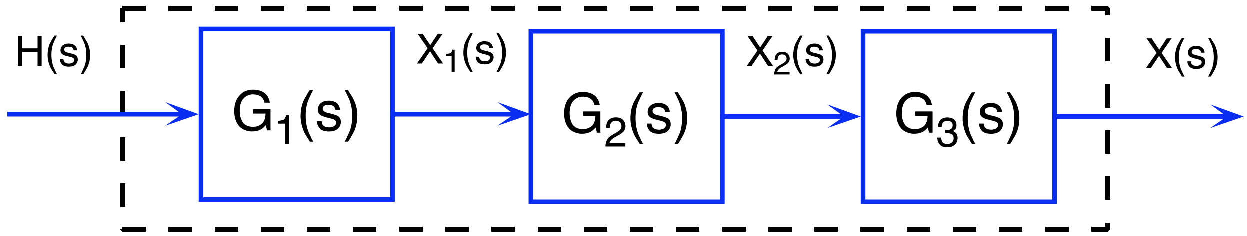


- Output = input to next
- Models "stringing" of components

- Example: stereo system



  - CD player→ amplifier→ speakers
  - CD player: $G_1(s) = \dfrac{X_1(s)}{H(s)}$
    - Input: *H(s)* from CD laser reader
    - Output: CD voltage $X_1(s)$
  - Power amplifier: $G_2(s) = \dfrac{X_2(s)}{X_1(s)}$
    - Input: CD output voltage $X_1(s)$
    - Output: amp voltage $X_2(s)$
  - Speakers: $G_3(s) = \dfrac{X(s)}{X_2(s)}$
    - Input: amp voltage $X_2(s)$
    - Output: sound, acoustic pressure $X(s)$



- Overall transfer function is product of block transfer functions:

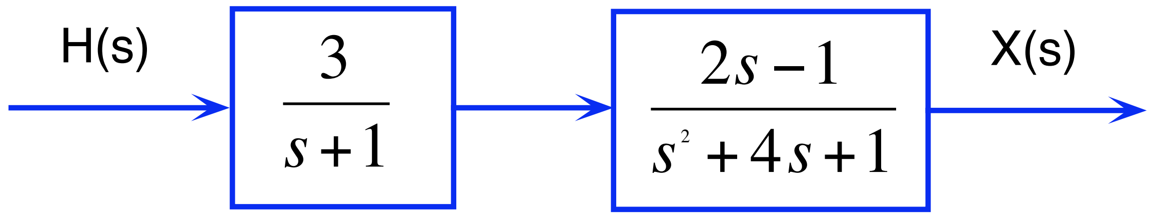
$$G(s) = \frac{X(s)}{H(s)} = \frac{X(s)}{X_2(s)} \frac{X_2(s)}{X_1(s)} \frac{X_1(s)}{H(s)} = G_1(s)G_2(s)G_3(s)$$

Note:

$$G_1(s) = \frac{X_1(s)}{H(s)}, \qquad G_2(s) = \frac{X_2(s)}{X_1(s)},$$

$$G_3(s) = \frac{X(s)}{X_2(s)}$$

Example



- Transfer function:

$$G(s) = \frac{X(s)}{H(s)} = G_1(s)G_2(s) = \frac{3}{s+1} \cdot \frac{2s-1}{s^2+4s+1}$$

$$G(s) = \frac{3(2s-1)}{(s+1)(s^2+4s+1)} = \frac{6s-3}{s^3+5s^2+5s+1}$$

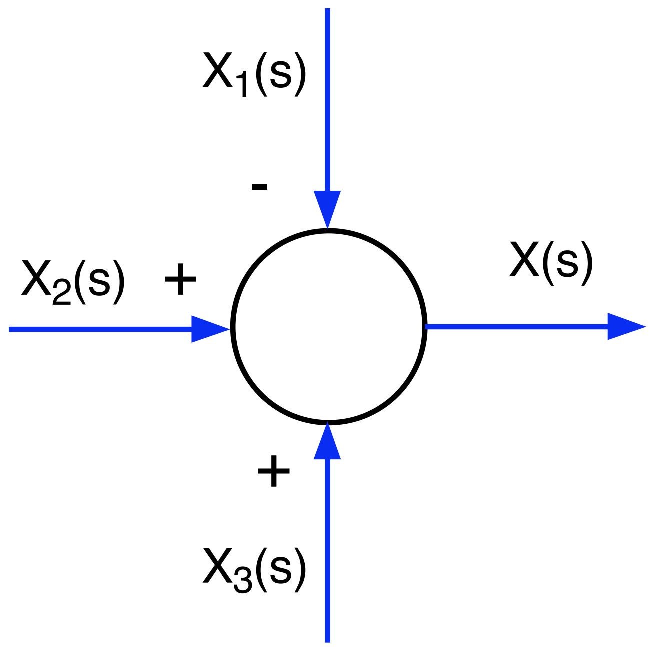
- <u>Poles</u> = roots of denominator (values of *s* such that transfer function becomes infinite)

$$p_1 = -1, \quad p_2, p_3 = -2 \pm \sqrt{3}$$

- <u>Zeros</u> = roots of numerator (values of *s* such that transfer function becomes 0)

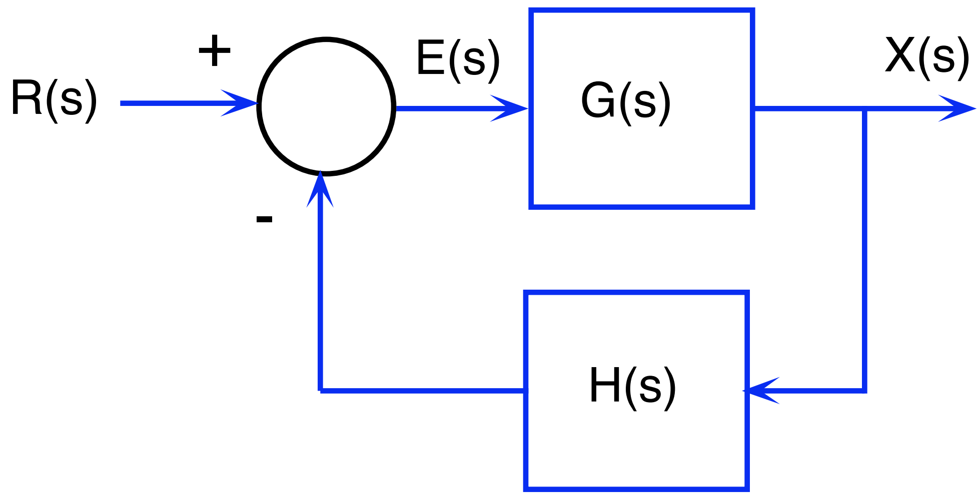
$$z_1 = 1/2$$

Summer (Summing Junction)



- Output = sum of inputs
- Sign on input => sign in equation
- Output $X(s) = - X_1(s) + X_2(s) + X_3(s)$

Example: Feedback Control System



- Goal: Closed loop transfer function

$$G_{cl}(s) = \frac{X(s)}{R(s)}$$

- Formulate:

$$E(s) = R(s) - H(s)X(s)$$
$$X(s) = G(s)E(s)$$

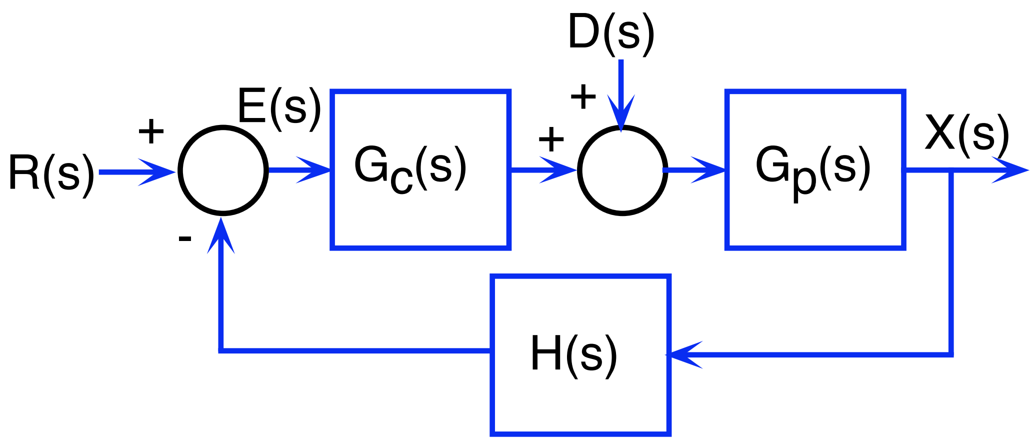
- Eliminate *E(s)*:

$$X(s) = G(s)E(s) = G(s)\left[R(s) - H(s)X(s)\right]$$

- Solve for closed loop *X(s)/R(s)*:

$$G_{cl}(s) = \frac{X(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Example: Feedback Controller with Disturbance



- Closed loop transfer function, reference input (temporarily set $D(s) = 0$)

$$G_{cl}(s) = \frac{X(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}$$

- Transfer function, disturbance (set $R(s) = 0$)

$$G_d(s) = \frac{X(s)}{D(s)} = \frac{G_p(s)}{1 + G_c(s)G_p(s)H(s)}$$

- Linear system, with both, sum outputs:

$$X(s) = G_{cl}(s)R(s) + G_d(s)D(s)$$

$$X(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}R(s) + \frac{G_p(s)}{1 + G_c(s)G_p(s)H(s)}D(s)$$