Stick-Slip

- Vibrations caused by friction interactions between sliding surfaces
- Cause of all sliding noise:
  - brake noise:
    - Squeal & squeak (high frequency: ~0.6–2 kHz)
    - Moan, groan, judder, chatter (low frequency: < 0.6 kHz)
  - Chalk squeal on board
  - Squeaky shoes on floor

Mechanism, Cause & Requirements

- Necessary conditions:
  - Friction coefficient $\mu$: static $\mu_s >$ kinetic $\mu_d$
  - Slope $\mu_d$ vs. velocity $v$, negative
- Mass of slider + support stiffness => vibration
- Negative slope => positive damping => instability
Model

- Slider mass $M$ against counter-surface
- Stiffness $k$ of slider support
- Normal force $W$
- Friction force $\mu W$, with negative slope

Physical model and bond graph

![Physical model and bond graph diagram]

- $W$
- $v_f$
- $M$
- $k$
- $v_h = p_h/M$
- $\mu(v_s) C \alpha^{3/2}$
- $\frac{dx}{dt}$
Equations of motion from bond graph:

- **Horizontal motions:**
  
  \[
  \frac{dp_h}{dt} = -kx + \mu(v_f - p_h/M) \cdot C \cdot \alpha^{3/2}
  \]

  \[
  \frac{dx}{dt} = p_h/M
  \]

- **Vertical motions:**
  
  \[
  \frac{dp_f}{dt} = C \cdot \alpha^{3/2}
  \]

  \[
  \frac{d\alpha}{dt} = p_v/M - p_f/M_f
  \]

  \[
  \frac{dp_v}{dt} = W - C \cdot \alpha^{3/2}
  \]
• Nonlinear contact stiffness $P = P(\alpha)$

$$P = C \alpha^{3/2}, \quad C = \frac{4}{3\pi(k_1+k_2)} \sqrt{\frac{R_1R_2}{R_1+R_2}}$$

• $\Rightarrow$ nonlinear contact vibrations
DRY FRICTION

- Dependent on
  - REAL contact area
  - Normal force $W$
  - Materials & environment
  - Weakly dependent on sliding speed $v$

\[ \mu(v) = \mu_k + (\mu_s - \mu_k) \exp\{-v/v_o\}\]

- For small $v$, Taylor series $\Rightarrow$

\[ \mu(v) \approx \mu_s - v(\mu_s - \mu_k)/v_o + (\mu_s - \mu_k)(v/v_o)^2/2 + \ldots \]
As sliding velocity \( v \) increases, friction coefficient \( \mu \) and friction force **decrease**

\[ \mu_s \quad v_0 \quad v \quad \mu_k \]

\( \Rightarrow \) Unstable behavior

- Root cause of friction induced noise!
• Substitute \( \frac{dx}{dt} = \frac{p_h}{M} \)

\[
dp_h/dt = -kx + \mu(v_f - \frac{p_h}{M}) C \alpha^{3/2}
\]

• \( M\ddot{x} + kx - \mu(v_f - \dot{x})C\alpha^{3/2} = 0 \)

• Substitute \( \mu(v) \approx \mu_s - \frac{v(\mu_s - \mu_k)}{v_o} \)

\[
\ddot{x} - \dot{x}\left(\frac{\mu_s - \mu_k}{v_o}\right) C \alpha^{3/2} + \frac{k}{M} x = \left[\mu_s - v_f\left(\frac{\mu_s - \mu_k}{v_o}\right)\right] C \alpha^{3/2}
\]

• Compare with second order system, standard form

\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f(t)
\]

• Damping term negative \( \Rightarrow \) unstable behavior @ low speed
Step response, 2nd order system

\[ x(t) = \begin{cases} 
\frac{f_o}{\omega_n^2} \left[ 1 + \frac{\xi - \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} e^{-\omega_n t \sqrt{\xi^2 - 1}} - \frac{\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} e^{-\omega_n t \sqrt{\xi^2 - 1}} \right], & \xi > 1 \\
\frac{f_o}{\omega_n^2} \left[ 1 - (1 + \omega_n t) e^{-\omega_n t} \right], & \xi = 1 \\
\frac{f_o}{\omega_n^2} \left[ 1 - e^{-\omega_n t} \left\{ \cos(\omega_n \sqrt{1 - \xi^2} t) + \frac{\xi}{\sqrt{1 - \xi^2}} \sin(\omega_n \sqrt{1 - \xi^2} t) \right\} \right], & 0 \leq \xi < 1 
\end{cases} \]
Vibration Response, Stick/Slip

- Let $W = C \alpha^{3/2}$, no vertical motions
- Horizontal motions governed by

$$M\ddot{x} + kx - \left[ \mu_k + (\mu_s - \mu_k)e^{-\left(\frac{v_f - \dot{x}}{v_o}\right)} \right]W = 0$$

Rearrange

$$\ddot{x} + \frac{k}{M} x - (\mu_s - \mu_k)e^{-\left(\frac{v_f - \dot{x}}{v_o}\right)} \frac{W}{M} = \mu_k \frac{W}{M}$$

Rearrange again, $g$ is gravity

$$\ddot{x} - (\mu_s - \mu_k)e^{-\left(\frac{v_f - \dot{x}}{v_o}\right)} g + \omega_n^2 x = \mu_k g$$
Simulation (Mathematica)

- Differential equation:

\[ \ddot{x} - (\mu_s - \mu_k) \left( \frac{v_f - \dot{x}}{v_o} \right) g + \omega_n^2 x = \mu_k g \]

- Parameter values:

\( \mu_s = 0.2, \ \mu_k = 0.1, \ v_o = 1 \text{ m s}^{-1}, \ \omega_n/2\pi = 1 \text{ kHz} \)

- Initial conditions: \( \dot{x}(0) = x(0) = 0 \)

- Self sustained oscillation at natural frequency!

- Larger amplitude @ smaller sliding speed
Simulation (Mathematica)

- Let \( V_v = p_v/M, \) \( V_h = p_h/M, \) \( V_f = p_f/M_f \)

\[ C_o = C/M, \quad m = M/M_f \]

\[ C = \frac{4}{3\pi(k_1+k_2)} \sqrt{\frac{R_1R_2}{R_1+R_2}} \]

\[ k_i = \frac{(1 - \nu_i^2)}{E_i} \]

- State equations become:

\[ \frac{dV_h}{dt} = -\omega_n^2 x + \mu(v_f - V_h) C_o \alpha^{3/2} \]

\[ \frac{dx}{dt} = V_h \]

\[ \frac{dV_f}{dt} = C_o m \alpha^{3/2} \]

\[ \frac{d\alpha}{dt} = V_v - V_f \]

\[ \frac{dV_v}{dt} = g - C_o \alpha^{3/2} \]

- Zero initial conditions
Parameter values:

\[ \mu_s = 0.2, \quad \mu_k = 0.1 \]

\[ E_i = 200 \text{ Gpa}, \quad \nu_i = 0.3 \]

\[ m = 0.001, \quad C_o = 4.63 \times 10^{10} \]

\[ v_o = 1 \text{ m s}^{-1}, \quad v_f = 0.001 \text{ m s}^{-1}, \quad \omega_n/2\pi = 1 \text{ kHz} \]

Self sustained oscillation at natural frequency!

Amplitude of horizontal motions grow, due to interaction with vertical motions