Inventory Theory

Inventory Models

This chapter will consider several model types. Supplement 2 describes deterministic models. Even though many features of an inventory system involve uncertainty of some kind, it is common to assume much simpler deterministic models for which solutions are found using calculus. Deterministic models also provide a base on which to incorporate assumptions concerning uncertainty. Supplement 3 adds a stochastic dimension to the model with random product demand. Supplement 4 begins discussion of stochastic inventory systems with the single period stochastic model. The model has applications for products for which the ordering process is nonrepeating. The remainder of the chapter considers models with an infinite time horizon and several assumptions regarding the costs of operation. Supplement.5 and .6 derive optimum solutions for the \((s, S)\) policy under a variety of conditions. This policy places an order up to level \(S\) when the inventory level falls to the reorder point \(s\). Supplement 7 extends these results to the \((R, S)\) policy. In this case the inventory is observed periodically (with a time interval \(R\)), and is replenished to level \(S\).

Flow, Inventory and Time

An inventory is represented in the simple diagram of Fig. 1. Items flow into the system, remain for a time and then flow out. Inventories occur whenever the time an individual enters is different than when it leaves. During the intervening interval the item is part of the inventory.

![Figure 1. A system component with inventory.](image)

Flow In

\(\text{Inventory Level (Residence Time)}\)

Flow Out

For example, say the box in Fig. 1 represents a manufacturing process that takes a fixed amount of time. A product entering the box at one moment leaves the box one hour later. Products arrive at a rate of 100 per hour. Clearly, if we look in the box, we will find some number of items. That number is the inventory level. The relation between flow, time and inventory level that is basic to all systems is

\[
\text{Inventory Level} = \text{(Flow Rate)} \times \text{(Residence Time)},
\]

where the flow rate is expressed in the same time units as the residence time. For the example, we have

\[
\text{Inventory Level} = (100 \text{ products/hour}) \times (1 \text{ hour}) = 100 \text{ products}.
\]
When the factors in this expression are not constant in time, the expression relates
time averaged quantities.

Whenever two of the factors in the above expression are given, the
third is easily computed. Consider a queueing system for which customers are
observed to arrive at an average rate of 10 per hour. When the customer finds the
servers busy, he or she must wait. Customers in the system, either waiting or be
served, are the inventory for this system. Using a sampling procedure we
determine that the average number of customers in the inventory is 5. We ask,
how long, on the average, is each customer in the system? Using the relation
between the flow, time and inventory, we determine the answer as 0.5 hours. The
relation receives extensive use in queueing analysis where it is called Little's
Law\(^1\).

The relation between time and inventory is important, because very
often the reducing the throughput time for a system is just as important as
reducing the inventory level. Since they are proportional changing one factor
inevitably changes the other.

The Inventory Level

The inventory level depends on the relative rates of flow in and out of the system.
Define \(y(t)\) as the rate of input flow at time \(t\) and \(Y(t)\) the cumulative flow into the
system. Define \(z(t)\) as the rate of output flow at time \(t\) and \(Z(t)\) as the cumulative
flow out of the system. The inventory level, \(I(t)\) is the cumulative input less the
cumulative output.

\[
I(t) = Y(t) - Z(t) = \int_{0}^{t} y(x)dx - \int_{0}^{t} z(x)dx
\]  \hspace{1cm} (2)

Fig. 2 represents the inventory for a system when the rates vary with time.

![Inventory Fluctuations](image)

Figure 2. Inventory fluctuations as a function of time.

The figure might represent a raw material inventory. The flow out of
inventory is a relatively continuous activity where individual items are placed into

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\(^1\) Little's Law is related to Queueing systems in Section 18.1
the production system for processing. To replenish the inventory, an order is placed to a supplier. After some delay time, called the lead time, the raw material is delivered in a lot of a specified amount. At the moment of delivery, the rate of input is infinite and at other times it is zero. Whenever the instantaneous rates of input and output to a component are not the same, the inventory level changes. When the input rate is higher, inventory grows; when output rate is higher, inventory declines.

Usually the inventory level remains positive. This corresponds to the presence of on hand inventory. In cases where the cumulative output exceeds the cumulative input, the inventory level is negative. We call this a backorder or shortage condition. A backorder is a stored output requirement that is delivered when the inventory finally becomes positive. Backorders are possible for some systems, while they are not for others. A finished product inventory, for example, may promise later delivery if a customer arrives to find no product available. Alternatively, a customer with alternative suppliers may go elsewhere and the sale is lost. In cases where backorders are impossible, the inventory level is not allowed to become negative. The demands on the inventory that occur while the inventory level is zero are called lost sales.

Variability, Uncertainty and Complexity

There are many reasons for variability and uncertainty in inventory systems. The rates of withdrawal from the system may depend on customer demand which is variable in time and uncertain in amount. There may be returns from customers. Lots may be delivered with defects causing uncertainty in quantities delivered. The lead time associated with an order for replenishment depends on the capabilities of the supplier which is usually variable and not known with certainty. The response of a customer to a shortage condition may be uncertain.

Inventory systems are often complex with one component of the system feeding another. Fig. 3 shows a simple serial manufacturing system producing a single product.

We identify planned inventories in Fig. 3 as inverted triangles, particularly the raw material and finished goods inventories. Material passing through the production process is often called work in process (WIP). These are materials waiting for processing as in the delay blocks of the figure, materials undergoing processing in the operation blocks, or materials undergoing inspection.
in the *inspection* blocks. All the components of inventory contribute to the cost of production in terms of handling and investment costs, and all require management attention.

For our analysis, we will often consider one component of the system separate from the remainder, particularly the raw material or finished goods inventories. In reality, rarely can these be managed independently. The material leaving a raw material inventory does not leave the system, rather it flows into the remainder of the production system. Similarly, material entering a finished goods inventory comes from the system. Any analysis that optimizes one inventory independent of the others must provide less than an optimum solution for the system as a whole.