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Variable Structure Model Synthesis
for Switched Systems

by

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Dissertation

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Variable Structure Model Synthesis
for Switched Systems

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S.V. Sreenivasan
To mom, dad, and Sonya.

To Claudia the most important woman in my life.

To Desi and Lucy.
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Abstract

Model reduction methods exist for a large class of linear systems. There even exist model deduction algorithms for certain classes of nonlinear systems that employ sinusoidal input describing functions to quasi-linearize the nonlinear elements. However, to date there does not exist a well established model synthesis methodology for physically switched systems. A method that borrows ideas from model reduction and variable structure system (VSS) theory is presented for synthesizing models of switched systems. Metrics are given for measuring the relative fidelity of these models. These ideas are then extended to develop an approach for doing more computationally efficient simulations of a vehicle model for virtual mission studies. Simpler sub-models of varying fidelity are derived for the transient and steady-state modes of the vehicle; each emphasize the dynamics of the system for each mode. VSS theory is used to switch between the sub-models. The simulation is improved by using reduced models for those modes of operation that require less detail to predict dynamics.
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Chapter 1

Introduction

Engineers have always been interested in methods to better synthesize models. Nowhere is that more evident than in the advent of annual conferences such as the *Automated Modeling for Design Conference* sponsored by ASME. The rapid development of personal computers and affordable workstations has proliferated the evolution of modeling and simulation. In recent years there has been increased interest in the systematic synthesis of complex system models. Particularly the synthesis of models that are tailored to specific tasks such as controller design, system analysis, real-time simulation, or component design.

In this dissertation we will focus on methods for synthesizing models of switched systems. Methods will be shown for synthesizing models for switched systems, and metrics for measuring the fidelity of these models will be presented.
1.1 Model Fidelity

Introduced here is the concept of variable fidelity modeling. Webster’s Dictionary defines “fidelity” as exactness or accuracy. It also refers to fidelity as the “degree of accuracy with which sound or images are recorded or reproduced.” In this dissertation, “fidelity” will generally refer to the degree of accuracy of a model. “Fidelity” may refer to the order of a model – how many physical states are tracked. A model of higher order will involve more states to track the dynamics of the system. It might also refer to the level of detail involved in modeling elements of the system. For instance, a diode in an electric converter can be modeled as an ideal switch or it can be modeled with more detail by employing the constitutive law for a pn-junction. Generally speaking, a high fidelity model would be one that involves intricate detail and more states to account for the system dynamics more accurately. More specifically, in this dissertation “fidelity” will refer to how accurately a model predicts the physical dynamics of the real system.

Variable fidelity modeling is an approach that recognizes that a single model is not best suited for all possible purposes (i.e. controller design, system analysis, real-time simulation, etc.). This approach recognizes that to better serve a given task a model’s fidelity must be tailored to more efficiently extract only the necessary information about the system’s physical behavior to sufficiently predict the system dynamics necessary to meet predefined objectives. Variable fidelity modeling employs sub-models of varying fidelity to better serve the user’s task.
1.2 Switched Systems

Systems that employ physically switched elements are particularly difficult to model. *Switches* can be thought of as physical mechanisms that discontinuously or abruptly redirect the flow of power in a system. Some switch-like mechanisms are semiconductor switches, hydraulic valves, and mechanical clutches. The switches introduce nonlinear dissipative effects and hard discontinuities which are more laborious to model and which can cause significant havoc for the average numerical integration routine.

*Switched systems* are systems that incorporate switch-like elements. More specifically, they are designed to employ a switching system structure for the purpose of efficient power conversion and/or to incorporate an inherent control structure. Currently, there does not exist a well established generalized method for systematically synthesizing models for switched systems – a methodology that would aid a modeler in deriving a model of fidelity or complexity well suited for his/her task.

This is the problem the author will explore throughout the course of this dissertation. A methodology will be presented that could lend considerable aid to the synthesis of reduced (i.e. more simplified) models for switched systems. Furthermore, metrics are presented for quantifying the relative fidelity of the synthesized model.

The material presented has potentially significant application in the design, simulation, and analysis of switched systems like electronic power converters, automatic transmissions, and valved pumps.
1.3 Background

This all began with one seemingly simple problem: design a better hydraulic-ram pump. The hydraulic-ram pump is used to pump water from a river or creek to a pond or water reservoir. It is used where conventional power supplies are not readily available. The pump requires no outside power to actuate its valves (refer to Figure 1.1); it simply uses the inflow of water to operate.

The hydraulic-ram pump consists of three primary elements: a waste-gate valve, 

a check valve, and a pumping chamber. The closing of the waste-gate valve depends on
the flow of water from the inlet. When the flow creates enough drag it shuts the valve
closed, the pressure builds up in the inlet junction, and the check valve cracks open
allowing water to flow into the pumping chamber where the pressure begins to build.
The pump cycles through this and boosts the pressure at the outlet.

The system is complicated by several factors. A few of which are:

- The flow in and out of the pump is generally turbulent making the dissipative
effects nonlinear.

- The valves are switch-like elements, but they do not operate like ideal switches.
  Their dynamic actuation is critical to the pump’s power conversion.

- The waste-gate valve has a hard stop that abruptly cuts the flow out of the valve.

- The inlet pipe and outlet hose introduce distributed-parameter effects.

- The specific pump under study in the lab vibrates because it is only secured to
  the ground by rigid clamps that provide no damping. This allows high frequency
  vibrations that affect the pressures in the pump.

At first glance, it is unclear how critical some of these complications are to modeling
the hydraulic ram pump. To model this system a minimum of 3 states is required. How-
ever, to include the distributed effects using a lumped-model approach, a finite number
of additional states are necessary. There are over 5 parameters that can be adjusted to
optimize the performance of the pump. Where does one start? What are the key parameters and the general trends? What can be neglected to simplify the model? How does one synthesize a model of the hydraulic ram of a fidelity that could predict general trends and narrow down many parameters to fewer more key parameters? More generally, how does one synthesize a model of a switched system – a system that employs physical switch-like mechanisms that change its structure and/or dynamic operation?

We have recognized that the hydraulic ram has an electric analog. It can be compared to the electronic boost converter (refer to Figure 1.2 (a)) – a power converter that performs a similar function in the electrical domain. The boost converter is also a useful problem for study because many proven methods exist for modeling electrical power converters. The methods for modeling power converters range from more intensively analytical state-variable methods that account for all the high frequency dynamics due to switching to more elegant averaged methods that produce faster yet accurate results [18, 28]. Typical power converter switches are operated at a frequency much higher than the overall system dynamics, leading to difficulties in the simulation due to stiffness. Averaged methods, particularly state-space averaging, are mathematical approaches designed to overcome this difficulty and have proven to be useful for predicting overall system dynamics of up to half the switch frequency [18, 28].

Key differences between the pump and boost converter make many of the more elegant methods for modeling electric power converters ill suited for modeling the hydraulic ram pump. The switches of the pump operate at a frequency of nearly the same
order of magnitude as the frequency of the overall system dynamics. Averaged methods would therefore not as accurately capture these dynamics. Furthermore, the semiconductor switches in electric power converters are typically designed to be very efficient and modeling the switches as ideal often proves to be more than adequate. The complicated actuation of the valves may not be so ideal. There is significant power dissipated by these valves. The more established techniques used for modeling electric converters have limitations that result in models that would not adequately capture both the transient and steady-state behavior of the pump. This investigation did provide insight as to what more was necessary to model the pump more appropriately.

A literature search on “model simplification” garnered an abundance of literature on model reduction algorithms. Model order reduction algorithms (MORAs) have been used for several decades and are well established. Frequency response methods are most common (e.g. Chen [5]) and involve more simple calculations. However, the results obtained using such methods do not guarantee a stable model [13]. There are time-domain methods like Davison’s [6] that preserve dominant eigenvalues of the system. These methods are mathematically more intensive. They do result in stable models but these models often fail to adequately approximate the original system [13]. Researchers have utilized a mixed approach (e.g. [20, 24]) to resolve the limitations of previous methods and in hopes of combining advantages of frequency-response and time-domain approaches. These methods are well established and provide a viable approach for model reduction of linear systems.
The hydraulic ram is nonlinear so existing methods for model reduction could prove to have limited applicability unless the system model can be linearized for certain frequency ranges of operation. Perhaps a more fruitful method is the model reduction approach introduced by Wilson et al. [25, 29, 30, 31]. The model order deduction algorithm for nonlinear systems (MODANS) uses a sinusoidal input describing function (SIDF) to quasi-linearize each nonlinear element. This method has been applied to model “serially-connected unidimensional, electro-mechanical systems” [25] and may or may not prove useful for handling the nonlinearities present in the hydraulic ram. The dynamics of the hydraulic ram include discontinuities that may not be adequately represented using SIDFs. Furthermore, in the investigations presented by Wilson et al. the system inputs are assumed to be sinusoidal. That is not the case with the pump. Further investigation is needed to determine the possible limitations of the MODANS approach when modeling switched systems.

A search for literature on modeling switch-like systems produced two relevant studies. The first by Bass [1] suggests that it might be useful to model switched power converters as variable structure systems (VSS) whose switching structures resemble sliding mode controls (SMCs). Utkin et al. [26] even presents a SMC approach for controlling electronic power converters. In this way these systems could be analyzed using methods borrowed from VSS theory, which is generally designed to handle nonlinear problems. A second paper by Liaw [13] introduced an interesting idea for model reduction. Liaw was motivated by the limitations of previous methods to produce a
better model reduction method. Liaw’s idea was to use two reduced-order models: one that emphasizes the transient response and a second that captures the steady-state dynamics. He used ad hoc switching functions to switch between the two models. In effect he had a variable structure system made up of two sub-models that capture the system dynamics over two different frequency ranges.

1.4 Challenges and Contributions

In order to establish a model reduction methodology that is applicable to switched systems, the following hypothesis if formed:

“A method that borrows ideas from model reduction and VSS theory can be used to model and analyze switched systems. It may also lead to metrics for synthesizing models of desired fidelity.”

VSS theory can be used to breakdown a “switched system” into its more basic variable structures. A model reduction technique could then be used to synthesize the submodels for each variable structure. Then, SMC ideas can be employed to represent the switching structure. In this way, one can garner insight into how the system operates while in each of its modes and how it switches between them. Such insight can be used to improve the design of the switched system. This method will allow one to make connections between the continuous dynamics of the individual structures and the overall discontinuous dynamics of the system.
The resulting work can be expanded to improve models of complex multi-power domain systems:

“Complex systems can be thought of in the framework of variable structures where each structure is a sub-model that emphasizes the system’s dynamics in a specific mode, time span, or frequency range. It is anticipated that the method developed can be used to create more efficient simulations that switch between system sub-models.”

For instance a system level model of a hybrid electric vehicle includes a myriad of sub-systems that undergo transient and steady-state dynamics depending on the operation of the vehicle. Model reduction methods can be used to derive sub-models that emphasize the vehicles dynamics during acceleration, cruising, braking, etc. The method implied here will be used to synthesize the sub-models and switch between them resulting in more efficient and perhaps significantly faster simulations that may be better suited for doing long-term mission studies of the vehicle.

These ideas present unique challenges that must be conquered in order to realize the proposed method. Current model reduction methods are designed to work for certain classes of problems. It must be determined if these techniques can be extended to model switched systems. If not, then a technique tailored to switched systems must be developed.
When synthesizing a model a metric is necessary for determining if the model is of a fidelity appropriate for the prescribed task. The MODA and MODANS methods presented by Wilson et al. involve a user-prescribed frequency range of interest (FROI). A similar metric could be employed for the method developed.

The following contributions were made in this doctoral research and are presented in this dissertation:

1. A methodology for synthesizing models of switched systems.
2. Metrics for measuring a switched-system model’s fidelity.
3. A variable fidelity modeling approach to produce more efficient mathematical representations that require less computational effort.

1.5 Dissertation Structure

Several test problems are presented here as case studies. They will be employed in the subsequent chapter to evaluate existing model reduction and model simplification techniques, their advantages and limitations. These problems will serve as a benchmark upon which the ideas and methodologies presented in the previous section will be developed and refined.

The systems to be examined are (refer to Figure 1.2):

- The electric boost converter
Each of these systems innately employ switch-like mechanisms to serve the purpose of providing a control structure and/or improving performance efficiency.

Existing methods will be more thoroughly explored in Chapter 2. In Chapter 3 the proposed methodology will be presented in detail and will be used to model the
three test cases. Presented in Chapter 4 are metrics for evaluating the fidelity of a model. These ideas will be expanded in Chapter 5 and used to do improved simulation and mission studies of a vehicle.

1.6 Summary

Some ideas have been presented for model reduction and complex system simulation, motivated by a modeling problem that has been investigated for some time now. The literature reviewed to date does not suggest how to best synthesize variable fidelity models for switched systems. However, the literature does provide a basis for possible areas of investigation. This research endeavors to provide improved methods for systematically modeling switched systems. Furthermore, the methods developed may prove useful in improving modeling and simulation of complex, multi-subsystem problems. In particular it may provide a more efficient means of simulating long-term mission studies for vehicles, where the vehicle undergoes dynamically different modes of operation.

This research has potential extended application for modeling and designing more complex systems like electric motor drives, differentials, and automatic transmissions. Also, it can be applied to systems that like the clutched yo-yo have discontinuous constraints. An example might be an articulated two-link robot manipulator designed to swing a racket for hitting tennis balls. The impact of the ball with the racket is a discontinuous constraint.
Chapter 2

Literature Review

Many existing methods provide well established formats for evaluated specific switching problems. These methods are detailed here to provide a basis for the ideas presented in this dissertation while motivating the development of this new approach. The test cases will be used in this chapter as a means for presenting and evaluating the application of several existing methods.

2.1 State-Space Averaged Modeling

During the 1970s engineers like Wester, Middlebrook and Čuk [18, 28] developed now well established methods for modeling the dynamic behavior of electronic power converters. The most common being the state-space averaged approach. Semiconductor switches are designed to have negligible parasitic losses and power electronic circuits
employing such devices are often highly efficient [10]. Furthermore, the frequency at which these semiconductor devices are switched on and off is typically several orders of magnitude higher than the overall system dynamic response. These characteristics allow one to make some simplifying assumptions that lead to an averaged approach. A thorough discussion of state-space averaging can be found in [7, 18, 19] et al. Averaging techniques in general involve writing separate state-space equations for the time interval that the controlled switch is “on” and the time interval that it is “off”:

\[
\begin{align*}
\dot{x} &= A_1 x + B_1 u \\
\dot{x} &= A_2 x + B_2 u \\
y_1 &= C_1 x + E_1 u \\
y_2 &= C_2 x + E_2 u
\end{align*}
\]  

(2.1)\\

where \(dT_s\) denotes the time interval that the switch is on, \(d'T_s\) denotes the interval that the switch is off, \(T_s\) denotes the switch period, and \(u\) is a vector denoting the voltage sources. Averaged methods have proven to sufficiently predict converter dynamics of up to half the switching frequency [7, 18, 19, 28]. Control of \(d\) is equivalent to control by pulse-width-modulation.

In state-space averaging, the separate sets of equations for each interval are replaced with a single averaged set that approximates the general small-signal dynamics over both intervals. This approximation is made possible because by design a DC-to-DC switching converter should have low output voltage ripple [7]. This is first accomplished by summing the equations for both intervals multiplied by the appropriate
fraction that represents the portion of the switching period that each set applies:

\[ \dot{x} = d(A_1 x + B_1 u) + d'(A_2 x + B_2 u) \quad \text{and} \quad y = d(C_1 x + E_1 u) + d'(C_2 x + E_2 u). \quad (2.2) \]

Note that Equation 2.2 is nonlinear because \( d(t) \) is multiplied by \( x \) and \( u \) which are generally functions of time. The small-signal result is obtained by perturbing Equation 2.2 about a steady-state operating point and simplifying:

\[
\frac{d}{dt} \hat{x} = A \hat{x}(t) + B \hat{u}(t) + [(A_1 - A_2)X + (B_1 - B_2)U] \hat{d}(t) + \text{H.O.T.} \\
\hat{y}(t) = C \hat{x}(t) + E \hat{u}(t) + [(C_1 - C_2)X + (E_1 - E_2)U] \hat{d}(t) + \text{H.O.T.} 
\]

where

\[
A = DA_1 + D'A_2 \\
B = DB_1 + D'B_2 \\
C = DC_1 + D'C_2 \\
E = DE_1 + D'E_2
\]

(2.4)

Here \( D, D', X \) and \( U \) represent the steady-state values of \( d(t) \), \( d'(t) \), the state vector, and input vector respectively and \( \hat{x}, \hat{y}, \hat{u}, \) and \( \hat{d} \) the perturbations about those steady state values. Because of the small-signal assumption the second-order nonlinear terms are negligible (i.e. \( \dot{x}(t) \hat{d}(t) \approx 0 \) and \( \dot{u}(t) \hat{d}(t) \approx 0 \)) and therefore are not shown here.

Take for example the boost converter shown in Figure 2.1. The state-space averaging approach can be readily applied to predict the overall averaged dynamics of the boost converter. State-space representations can be readily derived as follows:

\[
\begin{bmatrix}
\dot{\lambda} \\
\dot{q}
\end{bmatrix} =
\begin{bmatrix}
\frac{-R}{L} & 0 \\
0 & -\frac{1}{R_{\text{load}}C}
\end{bmatrix}
\begin{bmatrix}
\lambda \\
q
\end{bmatrix} +
\begin{bmatrix}
1 \\
0
\end{bmatrix} V_{in}
\]

(2.5)
Figure 2.1: (a) and (b) Boost converter. (c) and (d) BJT conducting. (e) and (f) Diode conducting.

when the BJT conducts, and

\[
\begin{bmatrix}
\dot{\lambda} \\
\dot{q}
\end{bmatrix} = \begin{bmatrix}
-\frac{R}{L} & -\frac{1}{C} \\
\frac{1}{T} & -\frac{1}{R_{\text{load}}C}
\end{bmatrix} \begin{bmatrix}
\lambda \\
q
\end{bmatrix} + \begin{bmatrix}
1 \\
0
\end{bmatrix} V_{\text{in}}
\] (2.6)

when the diode conducts. Applying the above averaging approach results in:

\[
\begin{bmatrix}
\frac{d}{dt} \dot{\lambda} \\
\frac{d}{dt} \dot{q}
\end{bmatrix} = \begin{bmatrix}
-\frac{R}{L} & -D' \frac{1}{C} \\
D \frac{1}{T} & -\frac{1}{R_{\text{load}}C}
\end{bmatrix} \begin{bmatrix}
\dot{\lambda} \\
\dot{q}
\end{bmatrix} + \begin{bmatrix}
1 \\
0
\end{bmatrix} \dot{V}_{\text{in}} + \begin{bmatrix}
0 & \frac{1}{C} \\
-\frac{1}{T} & 0
\end{bmatrix} \begin{bmatrix}
\Lambda \\
Q
\end{bmatrix} \dot{\lambda}
\] (2.7)

The state-space averaged approach is a good alternative for modeling the average dynamics of certain types of switched systems including electric converters.
2.2 Model Order Reduction Algorithms

*Model order reduction algorithms* (MORAs) are used to derive a reduced-order model from a higher-order model. These methods are commonly used for distributed-parameters systems. Common reasons for obtaining low-order models are [8]:

1. Low-order model simplify the understanding of a system.

2. They reduce computational effort during simulations.

3. Reduced-order models facilitate more simple controller laws because design of the controller is numerically more efficient.

The benefits of reduced-order models are better heuristic understanding with reduced computational/mathematical effort.

MORAs were developed to reduce models of linear time-invariant systems, and thus their application is limited to those systems that can be sufficiently represented by a linear model. A linear, time-invariant system can be represented as

\[
\dot{x} = Ax + Bu
\]

\[
y = Cx + Du
\]

The objective is to find a low-order model of the system of the form

\[
\dot{x}_r = A_r x_r + B_r u_r
\]

\[
y = C_r x_r + D_r u_r
\]
such that the output vector $y_r$ is best approximates (according to some prescribed criteria) the output vector $y$ of the original model [8]. There exist frequency-domain and time-domain algorithms. The basic idea behind a MORA is to use a metric like the dominant eigenvalues of $A$ to systematically remove elements from a model while maintaining a specified level of accuracy in the prediction of outputs $y$. These methods could be characterized as “top-down” approaches because they begin with a higher fidelity model and trim down to a reduced model.

Researchers such as Wilson and Stein [29, 30] took this idea a step further to create a “bottom-up” approach. The model order deduction algorithm was designed to synthesize, rather than reduce, mathematical models of systems with distributed elements. Based on a geometric description of the system, the algorithm “coordinates the transformation of a...one-dimensional, network-type system into the lowest-order (linear) model that contains all the system eigenvalues (poles) within a user-specified frequency range of interest (FROI)” [29, 30]. The algorithm is a frequency based technique that basically iteratively augments the model systematically until the synthesized model transfer function $G(j\omega)$ captures the modes within the FROI. The algorithm is limited to model those systems that can be sufficiently represented with a linear model. Because the it is linear, the resulting frequency-domain, transfer-function model can be transformed back to the time-domain.

The MODA approach was extended to a more general class of select nonlinear systems by Wilson and Taylor [25, 31]. The frequency-domain model order deduction
algorithm for nonlinear systems (FD-MODANS) uses a describing function approach
to quasi-linearize the nonlinear model. The basic premise behind a describing function
(DF) is to replace nonlinearities with a quasi-linear terms whose gain is a function of
input amplitude $a$. The quasi-linear model can be deduced in much the same manner
as the linear models in MODA. The inputs are assumed to be sinusoidal so that the
nonlinearities can be replaced with sinusoidal-input describing functions (SIDFs). In
this way an input amplitude dependent transfer function $G(j\omega, a)$ can be derived for the
system. The problem is that the transfer function must be updated every time the inputs
amplitudes change.

To date application of this method has been limited to “serially connected unidi-
dimensional electro-mechanical systems” [25, 31]. An example is the ATB1000 “bench-
mark” problem which is a surrogate gun-turret test-bed. The problem includes non-
linear discontinuities like stick-slip friction and backlash, plus the the gun-turret is
composed of several flexible components. These nonlinearities are a consequence of
physical limits and not the design of the system. In other words, stick-slip friction and
back-lash are not designed into the system.

These approaches are predominately frequency-domain methods and assume
that the system can be sufficiently represented by linear or quasi-linear models. They
are designed predominately for modal type problems that include distributed elements.
Even the FD-MODANS approach was developed to synthesize models for distributed
parameter problems in the presence of nonlinearities that can be sufficiently represented
by SIDFs.

2.3 Variable Structure System Theory

A variable structure system (VSS) is one in which the control is allowed to change its structure. It can switch from one to another member of a set of possible continuous control functions [11, 26, 27]. In this way the control can strike a balance between performance and stability. The problem is then to design a set of controls using appropriate parameters and switching logic to efficiently control a system [26]. Sliding mode control (SMC) has wide applications in nonlinear systems where switching phenomena is common and where classical linear control methods fail to produce adequate or desirable controls. Here is a common example of a VSS referred to as the half-proportional controller developed by V. Ferner and presented in [11]:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(a + \alpha k)x_1
\end{align*}
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-(a + \alpha k) & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

where

\[
\alpha = \begin{cases} 
1, & \text{if } x_1x_2 \geq 0 \\
0, & \text{if } x_1x_2 < 0
\end{cases}
\]

and where \( a \) and \( k \) are such that

\[
a < 1,
\]

\[
a + k > 1, \text{ and}
\]

\[
a \text{ and } k > 0.
\]
Note that the above system has been written in the form \( \dot{x} = A(\alpha)x \). It has been written in this form so that the eigenvalues may be calculated and the stability of the variable structures determined. (The eigenvalues of a matrix are the values \( s \) that satisfy \( \det(A - sI) = 0 \).) The eigenvalues of matrix \( A \) are determined as follows:

\[
\begin{vmatrix}
-s & 1 \\
-(a + \alpha k) & -s
\end{vmatrix} = s^2 + (a + \alpha k) = 0
\]

which has the solutions

\[
s_{1,2} = \pm \sqrt{-(a + \alpha k)}
\]

The roots are purely imaginary (i.e. \( \text{Re}s_1 = \text{Re}s_2 = 0 \)) and according to [11] correspond to a set of ellipses elongated along the horizontal or vertical axis. When \( \alpha = 1 \) (i.e. \( a + k > 1 \)) the ellipses are elongated along the vertical axis as shown in Figure 2.2 (a), and for \( \alpha = 0 \) (i.e. \( a + 0 < 1 \)) the ellipses are elongated along the horizontal axis as shown in Figure 2.2 (b).

The control is defined by two variable structures. The two structures correspond to a family of ellipses in the phase plane as shown in Figure 2.2 (a) and (b). Though stable, neither family of ellipses is asymptotically stable. Figure 2.2 (c) shows how, with the proper switch logic, the two structures can be combined to quickly approach a point near \( (x_1, x_2) = (0, 0) \) which corresponds with asymptotic stability. VSS is used here to switch between two structures, both operating in a limit cycle, to produce a system that approaches global asymptotic stability.

VSS theory was developed to design a switched control structure to improve
Figure 2.2: Asymptotically stable VSS consisting of two stable structures [26]

performance and stability. Switched systems are systems in which by design or control the system structure, rather than control, is switched to achieve a desired task. There exist obvious parallels between VSSs and switched systems that suggest that VSS theory can alternatively be used to design and model switched systems.
Chapter 3

Variable Structure Model Synthesis

The variable structure formulation used here to derive reduced order models for now is specific to switched systems – systems that due to their inherent switch-like mechanisms exhibit properties similar to those of variable structure systems. We first begin as in any VSS formulation by systematically reducing the complex system to its more basic variable structures. Many tools including VSS theory, state-space analysis, automatic control theory, limit-cycle analysis, etc. can be used to derive and understand the dynamics and system response for each structure. Also, once broken down into the more manageable variable and continuous structures, more established model reduction techniques can be applied to the variable structure sub-models to further reduce the overall system model. Once the dynamic response of each structure (or sub-model) is understood then one can determine how the system “switches” between structures and how the varying structures are related to the overall system response. Mapping
the overall system response over the response of the varying structures can provide vital insight to the intricacies and dynamic operation of the complex system. Once it is understood how the varying structures affect overall system performance one can determine key parameters and then suggest ways to enhance system performance by adjusting them. A more formal and detailed methodology is presented in the following section.

### 3.1 Model Synthesis Methodology

The methodology presented here is used in the subsequent sections to synthesize models for the boost converter, the hydraulic-ram pump, and the clutched yo-yo. This methodology provides a framework for defining the problem, synthesizing a model, analyzing the system modeled, and ascertaining the usefulness of the synthesized model.

The methodology used in the following sections is as follows:

1. **Define problem and specify objectives.** A problem should be well-defined, and the objectives must be clear. Prescribing the objectives will motivate how model synthesis will proceed. The objectives are used to specify the fidelity of the model synthesized in Steps 5 and 6.

2. **Identify switching elements.** Determine the physical elements that introduce discontinuities in the physical dynamics of the system.
3. **Make simplifying assumptions.** Make reasonable assumptions about the each switch element and its actuation that facilitate the synthesis of the variable structure sub-models. Use these assumptions to define how the switch elements are actuated. Make any additional justifiable assumptions about the overall system that will further aid in the model development and system analysis.

4. **Identify separable variable structures.** Because of the discontinuous nature of
the switch elements employed, a switched system has discontinuous dynamics but is made up of substructures with continuous dynamics. The system switches between these substructures. By identifying the switch elements and specifying their assumed actuation, the variable substructures that make up the complete system can be identified.

5. **Synthesize models for variable structures.** It is more feasible to compose models for each variable structure because each is continuous. Furthermore, more tools are readily available to analyze and model continuous systems. These tools can be used to determine how the system behaves while operating in each variable structure and make more intuitive connections between the dynamics of each structure and the overall dynamics of the system.

6. **Define switching surface and compose complete model.** Now that the variable structure sub-models have been developed, one can proceed to define how the system switches between these structures based on the above assumptions. Using the defined switching structure, the parts are pieced together to compose the complete system model.

7. **Simulate model and ascertain model fidelity.** The complete can be simulated to evaluate if the model is of desirable fidelity to meet the prescribed objectives. This will be explored further in Chapter 4.

8. **Determine if model meets prescribed objectives.** Once a model is synthesized
and simulated, it is necessary to ascertain if the objectives set in the first step are met. If it does not meet objectives, the fidelity of the model must be adjusted and the model re-synthesized returning us to Step 5. (Chapter 4 details metrics for adjusting model fidelity.)

9. **STOP.** The process is stopped once objectives are met.

The methodology is presented in algorithmic form in Figure 3.1. The algorithm summarizes the general organization of the model development in the following sections. This methodology shall be referred to as *Variable Structure Model Synthesis* (VSMS).

### 3.2 The Electric Boost Converter

As has been suggested by Bass et al. [1], because VSS theory was developed to design systems that have a changing control structure it seems only natural to apply the theory to model and analyze an electric boost converter. An electric converter takes advantage of a changing structure to provide efficient power conversion. A boost converter can be thought of as a system that switches between two distinct structures. There currently exist several very well developed methods for analyzing, modeling, and designing electric converters. The work in this section is included, however, to suggest how VSS theory can be used to model systems with varying structure for which systematic, analytical methods are not well developed (as is the case with the hydraulic pump examined in
The boost converter employs two semiconductors switches. The transistor is often controlled by a modulated current source, while the current through the diode is modulated by the voltage drop across the diode. In other words, the control to the transistor is external while the control for the diode is internal. For purposes of this exercise the transistor and diode will be treated as ideal. In the ideal case only one switch conducts at a time. Therefore, the boost converter takes on the two distinct structures shown in Figure 2.1. In a real boost converter the actual switches are usually designed to be very efficient and parasitic losses are therefore often negligible.

Several factors are paramount in designing and controlling a boost converter. The boost converter, as mentioned earlier in Chapter 2, is used as a DC-to-DC converter that increases output voltage relative to input. The switching structure results in “chatter” or voltage ripple in the output voltage waveform. By design, the output voltage ripple should be minimal and the output voltage approximately constant at steady state. The second factor is the control of the output voltage. By adjusting the duty cycle of the controlled switch (the BJT), one can control the output voltage level.

### 3.2.1 Converter Model Synthesis

A state space approach can be used to determine the governing equations that represent the two structures and examine the characteristics of each structure. (Recall that in state-space analysis, linear systems are generally represented using the linear equation
\( \dot{x} = Ax + Bu \). When the transistor conducts, the boost converter takes on the structure shown in Figures 2.1 (c) and (d) and has the following governing equations,

\[
\begin{bmatrix}
\dot{\lambda} \\
\dot{q}
\end{bmatrix} =
\begin{bmatrix}
-\frac{R}{L} & 0 \\
0 & -\frac{L}{R_{\text{load}}C}
\end{bmatrix}
\begin{bmatrix}
\lambda \\
q
\end{bmatrix} +
\begin{bmatrix}
1 \\
0
\end{bmatrix}V_{\text{in}}.
\] (3.1)

Note that the charge in the capacitor (and therefore the capacitor voltage) responds as a first order system. If we examine the eigenvalues of the parameter matrix \( A \) we can determine the stability of this structure, i.e.,

\[
\begin{vmatrix}
-\frac{R}{L} - s & 0 \\
0 & -\frac{1}{R_{\text{load}}C} - s
\end{vmatrix} = \left( \frac{R}{L} + s \right) \left( \frac{1}{R_{\text{load}}C} + s \right) = 0.
\] (3.2)

This system has the following characteristic roots,

\[
s_1 = -\frac{R}{L} \quad \text{and} \quad s_2 = -\frac{1}{R_{\text{load}}C}.
\] (3.3)

Since \( R, R_{\text{load}}, L, \) and \( C \) are all real positive numbers, the roots must be real negative numbers, which implies that this structure is asymptotically stable with two asymptotes if the roots are distinct or one asymptote if the roots are repeated. (For more information on stability of variable structure systems, refer to [11]). For this structure, by some simple steady-state analysis, one can determine that the steady-state inductor current \( i_L \) and capacitor voltage \( v_C \) would be \( V_{\text{in}}/R \) and 0 respectively. A phase diagram for this structure would therefore be centered at \((V_{\text{in}}/R, 0)\).

Similarly we can apply this analysis to the structure the converter takes on when the diode is conducting (Figure 2.1c). It is governed by the following equations,

\[
\begin{bmatrix}
\dot{\lambda} \\
\dot{q}
\end{bmatrix} =
\begin{bmatrix}
-\frac{R}{L} & -\frac{1}{C} \\
\frac{1}{T} & -\frac{L}{R_{\text{load}}C}
\end{bmatrix}
\begin{bmatrix}
\lambda \\
q
\end{bmatrix} +
\begin{bmatrix}
1 \\
0
\end{bmatrix}V_{\text{in}}.
\] (3.4)
The charge and capacitor voltage now respond as second order systems because they are no longer dependent on \( q \) only but now also \( \lambda \). The characteristic roots of this set of equations are now given by a standard second order form,

\[
\begin{bmatrix}
\frac{R}{L} - s & -\frac{1}{C} \\
\frac{1}{L} & -\frac{1}{R_{\text{load}}C} - s
\end{bmatrix}
\]

\[
s^2 + \left( \frac{R}{L} + \frac{1}{R_{\text{load}}C} \right) s + \frac{1}{LC} \left( \frac{R}{R_{\text{load}}} + 1 \right) = 0 \quad (3.5)
\]

\[
s_{1,2} = -\frac{1}{2} \left( \frac{R}{L} + \frac{1}{R_{\text{load}}C} \right) \pm \frac{1}{2} \sqrt{\left( \frac{R}{L} - \frac{1}{R_{\text{load}}C} \right)^2 - \frac{4}{LC}}.
\]

There exist three possible cases. In the first case there could be two real, negative, distinct roots if the following is true,

\[
\left| \frac{R}{L} - \frac{1}{R_{\text{load}}C} \right| > \frac{2}{\sqrt{LC}}.
\]

As mentioned earlier, this case is asymptotically stable and has two asymptotes. The second case would be repeated negative roots which occurs when

\[
\left| \frac{R}{L} - \frac{1}{R_{\text{load}}C} \right| = \frac{2}{\sqrt{LC}}.
\]

Repeated, negative, roots suggest an asymptotically stable structure with a single asymptote. The final case involves complex roots with negative real parts.

\[
\left| \frac{R}{L} - \frac{1}{R_{\text{load}}C} \right| < \frac{2}{\sqrt{LC}}.
\]

The complex case with negative real part suggests a globally asymptotically stable system. Analysis shows that the inductor current and capacitor voltage would approach
\[ V_{in}/(R + R_{load}) \text{ and } V_{in}/(R/R_{load} + 1), \text{ respectively, at steady-state. Therefore, the phase diagram for this structure would be centered at } (V_{in}/(R + R_{load}), V_{in}/(R/R_{load} + 1)) \].

The complete system can be represented as a single set of equations using a VSS formulation. For example,

\begin{align*}
\dot{\lambda} &= V_{in} - \frac{R}{L}\lambda - \alpha \left[ \frac{1}{C}q \right] \\
\dot{q} &= \alpha \left[ \frac{1}{L}\lambda \right] - \frac{1}{R_{load}C}q
\end{align*}

(3.6)

where

\[
\alpha = \begin{cases} 
0 \text{ when the transistor conducts} \\
1 \text{ when the diode conducts}
\end{cases}
\]

We now have sets of equations for each variable structure and for the complete system.

### 3.2.2 Converter Simulation and Experimental Results

By overlapping the phase diagrams for the two structures, one can determine if and how the system reaches a limit cycle. Furthermore, the switching surface becomes apparent.

If we assume for the sake of simplicity that all parameters are of unit value except the inductor resistance \( R \), which typically is at least an order of magnitude less than \( R_{load} \), then the boost converter would have the phase diagram shown in Figure 3.2. When the transistor conducts the system has distinct, negative, real roots and therefore the phase diagram of this structure is stable but with two asymptotes. Furthermore, the system has complex roots with negative real part when the diode conducts and is therefore globally...
Figure 3.2: Phase diagrams for each structure and complete system.

asymptotically stable. Notice that the separate phase diagrams for each structure are shown on the same figure. The system switches between structures and follows the characteristics of each structure until it reaches a limit cycle.

A good measure of the validity of a VSMS model is to compare it to a model derived using more conventional methods. We will compare results from a state-space averaged model to those from a VSMS model and later to experimental results for a boost converter with the following parameters (refer to Figure 2.1 for graphical repre-
Figure 3.3: Dynamic results for the averaged model and the VSMS model.

\[ V_{in} = 2.5 \text{ Volts} \quad L = 200 \text{ mH} \quad R = 20 \Omega \quad C = 38 \mu\text{F} \]

\[ R_{load} = 1 \text{ k}\Omega \quad f_{\text{switch}} = 200 \text{ Hz} \quad D = 50\% \]

where \( f_{\text{switch}} \) and \( D \) are the switch frequency and duty cycle. As has been stated, state-space averaged models are well suited for predicting converter dynamics that are up to half the switching frequency. The VSMS model will, on the other hand, predict the “chatter” or rapid switch dynamics resulting from the turning on and off of the BJT. As confirmed by the results displayed in Figure 3.3, the two models correlate very well. The state-space averaged model approximates the average dynamics predicted by the VSMS model.
If we compare state-plane results, we see even further distinction between the two models (refer to Figure 3.4). The averaged model predicts that the system is asymptotically stable while the VSMS model predicts how the system reaches a limit cycle. The VSMS model shows the switching surface. The state-space averaged approach results in a linear model that lends itself to more simple linear controls theory analysis, while the VSMS approach requires a familiarity with VSS theory and with nonlinear control theory. However, the VSMS model lends itself to limit cycle analysis and allows a more detailed analysis of the converter control structure. The differences may seem superficial at first, but these differences will become increasingly important in the next
section where an attempt will be made to use these models to redesign the converter.

![Graph showing comparison of models and experimental results.](image)

Figure 3.5: Comparison of models and experimental results.

As evidenced by steady-state data in Figure 3.5, both models correlate well with experimental results. If one thinks of the output voltage as a DC constant offset plus an AC perturbation, the state-space averaged model would be a prediction of the DC component of the overall signal. In comparison, the VSMS model is a prediction of the complete dynamics. The VSMS model closely predicts the switching dynamics of the experimental results. Furthermore, the peak voltage predicted by the VSMS model is within 1% of the actual peak and the model minimum voltage is within 2% of the actual minimum. Considering that the diode and BJT have been idealized, the model is
fairly accurate.

### 3.2.3 Converter Redesign

When designing a boost converter there are several desirable characteristics. First, since a boost converter is a DC-to-DC electric converter, it is desirable to limit the ripple of the output voltage. Second, the converter should be efficient; power dissipation should be at a minimum. Note that $R_{\text{load}}$ is not innate to the actual converter. It is a load resistance and therefore the only dissipation to minimize would be that due to the inductor resistance $R$. However, this is a characteristic that depends on how well the inductor is designed and manufactured. If we assume that the designer is only choosing from different inductors and wishes to design the boost converter and not the inductor itself then that leaves three key parameters that affect the performance of the boost converter – the switch frequency $f_{\text{switch}}$, the inductance $L$, and capacitance $C$. The duty cycle is varied to achieve the desired output voltage and so does not directly affect efficiency. The frequency, however, at which the BJT is switched on and off does affect the output voltage ripple.

When the transistor is conducting, the capacitor is discharging and has first order response, as suggested by the structure in Figures 2.1 (c) and (d), i.e.,

$$\dot{q} = -\frac{1}{R_{\text{load}}C}q \Rightarrow v_C(t) = \frac{q_0}{C}\exp\left(-\frac{1}{R_{\text{load}}C}t\right). \quad (3.7)$$

As suggested by the above solution, by increasing the capacitance one can reduce the
rate at which the capacitor discharges, helping to reduce the output voltage ripple. Increasing the capacitance also has an effect on how fast the capacitor charges when the diode is conducting. The capacitor will charge slower also helping to reduce the output voltage ripple. The down side is that the overall system will reach steady-state slower during the transient, but since this is a DC-to-DC converter it will operate at steady-state and the transient effects will be unimportant.

The inductance $L$ does not come into play when the transistor is conducting and the capacitor is discharging. It only affects the capacitor voltage when the diode is conducting (refer to Figure 2.1 (e) and (f)) and the capacitor is charging. When the diode conducts the converter responds like a damped second order system,

$$\ddot{q} + \frac{RR_IC + L}{RLC} \dot{q} + \frac{R + RL}{RLC} q = \frac{V_{in}}{L},$$

with a natural frequency $\omega_n$ of

$$\omega_n = \sqrt{\frac{R + RL}{RLC}},$$

and a damping ratio $\zeta$ of

$$\zeta = \frac{L + RR_IC}{2\sqrt{(R + RL)RLC}}.$$

By adjusting the inductance $L$ and/or the capacitance $C$, the natural frequency at which the capacitor charges when the diode conducts can be changed. The natural frequency of the response can be decreased by increasing the inductance and/or the capacitance. To reduce the output voltage ripple, it would be desirable for the second order system to
respond at a natural frequency much lower than the switching frequency of the BJT so that the capacitor voltage will charge and discharge relatively slowly between cycles. This can be accomplished by increasing \( L \) and/or \( C \) to reduce the natural frequency of the system, or by increasing the switch frequency \( f_{\text{switch}} \) to give the capacitor less time to charge and discharge between cycles.

![Simulation results for output voltage (across the capacitor) for various capacitances.](image)

Figure 3.6: Simulation results for output voltage (across the capacitor) for various capacitances.

Since the capacitance \( C \) affects the output voltage in both cases (i.e. Equations 3.1 and 3.4), it would seem prudent to make changes there first (instead of the inductance \( L \)) to reduce the output voltage ripple.
Figure 3.7: Experimental results for converter using two different capacitances.

Let us return to the original model and again assume for simplicity that all parameters except $R_{load}$ and $C$ are of unit value. The inductor resistance is at least an order of magnitude less than the load and the capacitance is varied to explore the effects on output voltage ripple. Figure 3.6 shows the boost converter simulation results for two capacitances and switch frequencies. Note that the voltage ripple decreases as the capacitance is increased or the switch frequency is decreased. The model results are further confirmed by experimental data taken from the boost converter used for the above experimental results. The capacitance was increased by nearly an order of magnitude from $38 \mu F$ to $364 \mu F$. The average remained approximately the same but the
voltage ripple decreased significantly as evidenced by experimental data in Figure 3.7.

It has been shown how the VSMS model for a boost converter is derived, and how it can be used to redesign the converter. Converter design is a well established practice with many well developed tools readily available, but this is not the case for other more complex switched systems, such as the hydraulic-ram pump analyzed in the next section.

3.3 The Hydraulic-Ram Pump

The main motivation of the previous section (§3.2) was to present a formulation that could lend insight to the much more difficult problem of modeling and designing the hydraulic-ram pump. The purpose of this section is to suggest how the VSS formulation presented in previous sections can be expanded to derive simplified models of complex nonlinear systems like the hydraulic-ram pump. To accurately model the nonlinear dynamics of the pump, a higher fidelity model with many energetic states was initially employed. The initial model was rather intractable, and, despite much effort, simulations did not match experimental results even by order of magnitude. Reduced order models are more suitable for providing insight into complex systems – insight that can be used to improve design of such complex nonlinear systems. The purpose of modeling the pump is to sufficiently predict general trends so as to be able to suggest an improved redesign.
Figure 3.8: (a) and (b) Hydraulic-ram pump. (c) and (d) Waste-gate valve open and check valve closed. (e) and (f) Waste-gate valve closed and check valve open.

The pump requires no external input power aside from gravitational energy. It is used where electric power is not available to pump water from a stream or lake to a reservoir or water tank where it can be used for irrigation or other purposes. A well designed pump should sufficiently boost output pressure relative to the input to transport water to an elevated reservoir if necessary. Furthermore, the more volume of water it transports the better. Note that when the waste-gate valve is open water escapes the pump (refer to Figure 3.8). That water does not make it into the reservoir. A
well designed pump would provide significant pressure boost while limiting the flow of water out the waste-gate valve to maximize the transfer of water volume to an elevated tank or a water reservoir.

As already mentioned, the hydraulic-ram pump has many similarities with the boost converter that make it a hydraulic analog of the converter. (Historically, the hydraulic-ram pump came first [14].) In place of a BJT the hydraulic pump uses a waste-gate valve, and in place of the diode the pump uses the hydraulic analog – a check valve. There are some important differences between the two that make modeling the pump less trivial than modeling the converter. The boost converter and the hydraulic-ram pump differ in some key areas:

- **Switch Modulation:**

  - *Boost Converter:* The converter switches actuate much like ideal switches. The boost converter BJT is controlled by an outside source.

  - *Hydraulic-Ram Pump:* The pump uses a waste-gate valve that is modulated by the flow of water around the valve that creates a drag force to close the valve. The resulting valve motion is not ideal because there are significant periods over which both valves are partially open.

- **Switch Frequency:**

  - *Boost Converter:* The BJT and diode actuate at a frequency on the order of a kHz.
– *Hydraulic-Ram Pump*: The drag force opens and closes the valve at a frequency closer to 1-2 Hz.

- **Resistive Element Linearity:**
  
  – *Boost Converter*: The resistive elements in the boost converter have linear constitutive relations.

  – *Hydraulic-Ram Pump*: The flows through the pipes are impeded by a nonlinear resistances.

- **Energetic Element Distribution:**
  
  – *Boost Converter*: The energy storage in the pump can be accurately quantified by two lumped parameters – the inductance $L$ and the capacitance $C$.

  – *Hydraulic-Ram Pump*: There exists some compliance in the hose which accounts for some higher order dynamics. Additionally, the inlet pipe and outlet hose introduce distributed parameter effects. In some cases, to operate, the pipe may rely on these higher order dynamics.

Though there are some fundamental differences, the key similarities point to a possible reduced-order model. Experimental results showed some lower-order dynamics that could be approximated using a reduced order model that treats the hydraulic valves as if they opened and closed near ideally as is the case with the boost converter.
By assuming that the valves are open simultaneously for only a short period of time, the valves can be treated more ideally, and variable structure models can be derived to aid in understanding the operation of the pump. These models form the basis for an initial reduced-order model. This model should be able to predict the lower-order dynamics of the system. From there, the complexity of the model can be incremented to capture more complex dynamics. The reduced-order model, however, is more tractable and suitable for suggesting design improvements. Using the reduced model, one can key in on the major parameters that most affect the performance of the system.

3.3.1 Pump Model Synthesis

The hydraulic-ram pump can be broken down into more basic structures in much the same way the boost converter was in the previous section. If the waste-gate valve and check valve were to operate ideally then they would actuate like the BJT and diode in the boost converter. Since the object of this section is to derive a reduced-order model of the pump, a good place to start is by treating the valves as ideal switches. This allows one to identify the distinct variable structures of the pump system model. If we assume that only one valve flows water at a time then we can identify two distinct structures and their corresponding governing equations.

When the waste-gate valve is open and the check-valve closed, as shown in
Figure 3.8 (b), the system can be represented using the following equations,

\[
\dot{\Gamma}_{in} = P_{in} - \frac{R_{inlet}(\Gamma_{in})}{I_{inlet}} \Gamma_{in},
\]

\[
\dot{V}_{pc} = -\frac{1}{I_{outlet}} \Gamma_{out}, \quad \text{and}
\]

\[
\dot{\Gamma}_{out} = \frac{1}{C_{pc}} V_{pc} - \frac{R_{outlet}(\Gamma_{out})}{I_{outlet}} \Gamma_{out} - P_{out},
\]

(3.11)

where \(\Gamma_{in}, V_{pc},\) and \(\Gamma_{out}\) are the fluid momentum in the inlet pipe, the volume in the pumping chamber, and the fluid momentum in the outlet hose. In much the same manner, the governing equations can be derived for the system when the waste-gate valve is closed and the check valve is fully open (refer to Figure 3.8 (c)), i.e.,

\[
\dot{\Gamma}_{in} = P_{in} - \frac{R_{inlet}(\Gamma_{in})}{I_{inlet}} \Gamma_{in} - \frac{1}{C_{pc}} V_{pc},
\]

\[
\dot{V}_{pc} = \frac{1}{I_{inlet}} \Gamma_{in} - \frac{1}{I_{outlet}} \Gamma_{out}, \quad \text{and}
\]

\[
\dot{\Gamma}_{out} = \frac{1}{C_{pc}} V_{pc} - \frac{R_{outlet}(\Gamma_{out})}{I_{outlet}} \Gamma_{out} - P_{out}.
\]

(3.12)

Note that, in general, the resistive effects of the inlet and outlet pipes are a function of the flow.

Assuming the valves operate ideally, the complete system can be represented as a single set of equations using a VSS approach:

\[
\dot{\Gamma}_{in} = P_{in} - \frac{R_{inlet}(\Gamma_{in})}{I_{inlet}} \Gamma_{in} - \alpha \left[ \frac{1}{C_{pc}} V_{pc} \right],
\]

\[
\dot{V}_{pc} = \alpha \left[ \frac{1}{I_{inlet}} \Gamma_{in} \right] - \frac{1}{I_{outlet}} \Gamma_{out}, \quad \text{and}
\]

\[
\dot{\Gamma}_{out} = \frac{1}{C_{pc}} V_{pc} - \frac{R_{outlet}(\Gamma_{out})}{I_{outlet}} \Gamma_{out} - P_{out},
\]

(3.13)

where

\[
\alpha = \begin{cases} 
    0 & \text{when only the waste-gate valve is open} \\
    1 & \text{when only the check valve is open}
\end{cases}
\]
We now have sets of equations for each variable structure and for the complete system.

If the fluid flow through the pipe is laminar, the losses in the pipes can be treated as linear and a state-space approach much like that used to examine the boost converter can be applied to determine the phase diagrams for the various structures. The flow in this pump is generally turbulent and losses through the pipes are nonlinear. The pressure loss $\Delta P_{\text{loss}}$ for a circular pipe of length $L$, diameter $D$, and cross-sectional area $A$ is [9]

$$\Delta P_{\text{loss}} = \frac{1}{2} f(Re) \frac{L Q}{D A} \left| \frac{Q}{A} \right|$$

where $Q$ is the volumetric flow-rate and $f$ is the friction factor which depends on Reynolds number (and possibly surface roughness). The constitutive relation for the pressure drop is generally nonlinear. The volumetric flow-rate through the inlet pipe $Q_{\text{in}}$ can be expressed in terms of fluid momentum $\Gamma_{\text{in}}$,

$$Q_{\text{in}} = \frac{\Gamma_{\text{in}}}{I_{\text{inlet}}},$$

where $I_{\text{inlet}}$ is the fluid inertia and is calculated using the fluid density $\rho$,

$$I_{\text{inlet}} = \frac{\rho L}{A}.$$

The loss in the outlet hose can be treated in much the same manner. VSMS led to two continuous linear substructures for the boost converter. For the hydraulic ram pump, VSMS results in two continuous but nonlinear variable structures. By linearizing the system equations about an equilibrium point, it can be determined how the system will
behave near that point. Using a small-signal approximation about the origin results in the linear model

\[
\begin{bmatrix}
\frac{d\hat{\Gamma}_{in}}{dt} \\
\frac{d\hat{V}_{pc}}{dt} \\
\frac{d\hat{\Gamma}_{out}}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & -\alpha \left( \frac{1}{C_{pc}} \right) & 0 \\
\alpha \left( \frac{1}{I_{inlet}} \right) & 0 & -\frac{1}{I_{outlet}} \\
0 & \frac{1}{C_{pc}} & 0
\end{bmatrix} \begin{bmatrix}
\hat{\Gamma}_{in} \\
\hat{V}_{pc} \\
\hat{\Gamma}_{out}
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
P_{in} \\
P_{out}
\end{bmatrix}
\]

(3.14)

where \(\alpha\) is as defined in Equation 3.13 and where \(\hat{\Gamma}_{in}, \hat{V}_{pc},\) and \(\hat{\Gamma}_{out}\) are the perturbations about the origin. The origin is used because it is an equilibrium point for both variable structures. Because of the small-signal approach all nonlinear terms fall out, including those due to the inlet pipe and outlet hose resistances.

The linearized equations have been written in the form \(\dot{x} = Ax + Bu\) to aid in this analysis. The stability can be determined from the eigenvalues of matrix \(A\). When the waste-gate valve is completely open (i.e. \(\alpha = 0\)) the system is decoupled suggesting a degenerate case:

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -\frac{1}{I_{outlet}} \\
0 & \frac{1}{C_{pc}} & 0
\end{bmatrix}.
\]

The resulting matrix is not of full rank suggesting a degenerate case. If the characteristic curves for this structure are mapped in three dimensions, one plane will look like straight, parallel lines with a negative slope if the structure is stable. If the waste-gate valve is held open for a long time the pumping chamber would empty and the flow out the valve should reach a constant flow. In other words, this structure is stable.

The second structure corresponds to when the check valve is completely open,
or $\alpha = 1$. Matrix $A$ is then,

$$
A = \begin{bmatrix}
0 & -\frac{1}{c_{pc}} & 0 \\
\frac{1}{t_{inlet}} & 0 & -\frac{1}{t_{outlet}} \\
0 & \frac{1}{c_{pc}} & 0
\end{bmatrix}.
$$

The eigenvalues for this matrix are found from

$$
\begin{vmatrix}
    s & \frac{1}{c_{pc}} & 0 \\
    -\frac{1}{t_{inlet}} & s & \frac{1}{t_{outlet}} \\
    0 & -\frac{1}{c_{pc}} & s
\end{vmatrix} = s \left[ s^2 + \frac{1}{c_{pc}} \left( \frac{1}{t_{inlet}} + \frac{1}{t_{outlet}} \right) \right] = 0
$$

where

$$s_{1,2} = \pm \sqrt{-\frac{1}{c_{pc}} \left( \frac{1}{t_{inlet}} + \frac{1}{t_{outlet}} \right)} \text{ and } s_3 = 0.
$$

The first two eigenvalues are purely imaginary ($\text{Re}s_1 = \text{Re}s_2 = 0$) and the imaginary parts are of opposite sign ($\text{Im}s_2 < 0 < \text{Im}s_1$). In other words, the system has complex conjugate poles with zero real part. If the characteristic curves for this structure are mapped in three dimensions, one state plane will appear as concentric ellipses.

### 3.3.2 Pump Simulation and Experimental Results

The reduced-order model of the pump includes three states. The governing equation for $\Gamma_{ouu}$ remains the same in all three systems of equations (Equations 3.11 - 3.13). A three-dimensional plot would be necessary to produce the phase diagram of the VSMS model. However, it is useful to map the characteristics in the $P_{pc} - Q_{in}$ plane. Note that the pumping chamber pressure $P_{pc}$ is determined directly from the volume in the
Figure 3.9: Phase diagram for the hydraulic-ram pump using nonlinear variable structure models.

pumping chamber \((P_{pc} = V_{pc}/C_{pc})\). For the sake of simplicity only the characteristics corresponding to \(Q_{out} \mid r=0 = 5\) GPM are plotted (i.e. only \(\Gamma_{in} \mid r=0\) and \(V_{pc} \mid r=0\) are varied). (This was chosen because it corresponds to the steady-state outlet flow.) At steady-state the system dynamics closely follow the characteristic curves plotted for the two variable structures. The parameters used for the simulations in this and the following pump sections are detailed in Appendix A (§A.1 and §A.2).

The characteristic curves mapped in Figure 3.9 are plotted using the nonlinear variable structure models (Equations 3.11 - 3.13). The curves plotted in Figure 3.10
Figure 3.10: Phase diagram for the hydraulic-ram pump using linearized variable structure models.

are done so using the linearized models for the variable structures (Equation 3.14). The nonlinear VSMS model is mapped over these characteristics in both plots. Close examination is necessary to decipher the difference between the two plots. It is interesting to note that the nonlinear model does follow the general trend predicted by the linearized variable structure models. For instance, the characteristic curves corresponding to the system when the waste-gate valve is fully open are approximately straight, parallel lines with negative slope. Additionally, as predicted by the above analysis, the set of curves characteristic of the system when the check-valve is fully open are approximately con-
centric ellipses. The above analysis was therefore useful in predicting the general shape of the state trajectory. The nonlinear variable structure characteristic curves do better match the curvature of the pumps nonlinear model results. However, for only slightly less accurate results, the more simple linearized model provides favorable prediction of the pump’s state trajectory.

![Experimental setup for hydraulic-ram pump.](image)

Figure 3.11: Experimental setup for hydraulic-ram pump.

The pump dynamics are not well documented and to date have not been as extensively analyzed as the boost converter. A widely accepted model does not exist as is the case with boost converter. The state-space averaged model developed by Middlebrook et al. [7, 18, 19, 28] is well established and accepted. To validate the VSMS approach it is necessary to compare model results with experimental data. Much effort, over the past 3 years, was spent to validate the pump models. A commercial
hydraulic-ram pump was purchased and setup in the lab.\textsuperscript{1} Pressure transducers were used to monitor the pressures at junctions 1 and 2. Some time was spent in attempting to capture the inlet flow dynamics using a paddle wheel flow sensor. Several different flow sensors were considered. A *linear variable differential transformer* (LVDT) monitored the waste-gate valve vertical motion. To facilitate the model developed in the previous section, it was assumed that the valves actuate ideally. This was motivated by results garnered with the LVDT (refer to §A.3 and Figure A.2). Appendix A, §A.3, provides further details on the experimental setup and data collection.

The experimental data was collected and compared with the simulation results. Below is included the experimental data for the pump. For initial simulation results the valves are treated as ideal switches. Any dissipation due to the valves is initially ignored. As is detailed in §A.3 the flow sensor could not sufficiently capture the flow dynamics, so flow results are not shown.

Figure 3.12 shows experimental data and model predictions. For these initial results the pumping chamber was completely filled with water. Therefore, the compliance is due to the water compression, chamber wall expansion, and outlet hose compliance (refer to §A.1 for details). Note that the reduced-order model predictions generally follow the overall low-order dynamics of the actual pump experimental data. Interestingly, except for the initial spike, the simulation results peak at near the same magnitude and

\textsuperscript{1}The pump purchased is a Fleming Hydro-Ram. More detailed information on the pump can be found at The Ram Company web site: www.theramcompany.com.
Figure 3.12: Experimental data and reduced-order simulation results for waste-gate valve (WGV) motion junction pressures ($P_{jnc1}$ and $P_{jnc2}$)
time as the experimental data. The simulation results and experimental data differ for several reasons. First of all, the actual system exhibits several higher-order phenomena that are not accounted for in the simulation. For instance, the actual pump vibrates while operating, even though it is rigidly mounted to the ground. Furthermore, the inlet pipe and outlet hose are relatively long and may introduce significant distributed effects. The only resistances considered in the model are due to the inlet pipe and outlet hose. The experimental setup includes several fittings that introduce dissipative effects ignored in the reduced model. The valves also impede the flow of water through
them, and they have dynamics associated with them. Those dynamics are neglected in the reduced model. Including the dynamics of the valves introduces more energetic states into the system model. Despite neglecting effects that exist in the pump, the reduced model captures the critical low-order dynamics and produces correct order-of-magnitude results. These results reflect the pump performance that we seek to modify in a design process.

3.3.3 Pump Redesign

By making some simplifying assumptions, one is able to develop the VSMS pump model (Equations 3.11 - 3.13) and its linearized counterpart (Equation 3.14) detailed in §3.3.1. As mentioned earlier, a desirable pump design would maximize the total water volume transferred and/or the pressure in Junction 2. Taking a close look at matrix $A$ in Equation 3.14, one notices that the key parameters affecting pump dynamics are $\alpha$, $I_{inlet}$, $C_{pc}$, and $I_{outlet}$. However, $I_{inlet}$ and $I_{outlet}$ are not specifically related to the pump design, but rather to the inlet pipe and outlet hose respectively. This leaves $\alpha$ and $C_{pc}$ as the key parameters.

The VSMS model can be used to suggest general trends for key parameters. The model can be used to quickly determine how the pumping chamber capacitance affects the total volume of water transported out of the pump. For instance, Figure 3.13 shows the average flow rate out $Q_{out}$ of pump and the average junction 2 pressure $P_{jnc2}$. Both are plotted in reference to changes in the pumping chamber capacitance.
Figure 3.13: Performance parameters as a function of changes in pumping chamber capacitance ($C_{pc}$).

where ($C_{pc}$)$_{initial}$ is the capacitance used for the initial model. The simulation was run iteratively for pumping chamber capacitances that vary between 5% to 500% of the initial value. Also plotted are the average flow ratio $Q_{out}/Q_{in}$, average pressure boost $P_{jnc2}/P_{jnc1}$, average power efficiency $(P_{jnc2}Q_{out})/(P_{jnc1}Q_{in})$, and average output power $P_{jnc2}Q_{out}$. Results suggest that the output flow rate could be maximized if the pumping chamber capacitance is doubled while the junction 2 pressure is maximized if it is tripled. By maximizing the output power $P_{jnc2}Q_{out}$, a good balance can be achieved be-
tween maximizing the output flow rate and maximizing the junction 2 pressure. Though the power efficiency is maximized at about 500% of the pumping chamber capacitance, the output power drops. Since, efficiency is not paramount (we do not care if water is lost and flows back into the creek) the output power should be maximized to improve the pump design.

Steady-state analysis of the linearized model lends further insight. The following steady state problem can be evaluated to determine general trends. From Equation 3.14

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \left( \frac{1}{C_{pc}} \right) & 0 \\ \alpha \left( \frac{1}{I_{inlet}} \right) & 0 & -\frac{1}{I_{outlet}} \\ 0 & \frac{1}{C_{pc}} & 0 \end{bmatrix} \begin{bmatrix} (\Gamma_{in})_{ss} \\ (V_{pc})_{ss} \\ (\Gamma_{out})_{ss} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} P_{in} \\ P_{out} \end{bmatrix}$$

which is evaluated to derive the following steady state values,

$$(V_{pc})_{ss} = \frac{1}{\alpha} C_{pc} P_{in} = C_{pc} P_{out} \text{ and } \left( \frac{\Gamma_{out}}{\Gamma_{in}} \right)_{ss} = \alpha \left( \frac{I_{outlet}}{I_{inlet}} \right).$$

These results suggest that the steady-state volume of the pumping chamber $(V_{pc})_{ss}$ and hence Junction 2 pressure $(P_{jnc2} = \frac{(V_{pc})_{ss}}{C_{pc}})$ increase with pumping chamber capacitance supporting the results shown in Figure 3.13.

The flow rate can be directly related to fluid momentum as suggested earlier. Therefore the steady-state flow ratio can be predicted using the above results,

$$\left( \frac{Q_{out}}{Q_{in}} \right)_{ss} = \left( \frac{\Gamma_{out}}{\Gamma_{in}} \right)_{ss} \left( \frac{I_{inlet}}{I_{outlet}} \right) = \alpha \left( \frac{I_{outlet}}{I_{inlet}} \right) \left( \frac{I_{inlet}}{I_{outlet}} \right) = \alpha.$$

Recall that $\alpha$ is either 0 or 1 as defined above, and that the valves were assumed to open and close ideally. Therefore the duty cycle would be a measure of the percentage
of time during each switch period that the check valve is open. As the duty cycle is increased the check valve is open ($\alpha = 1$) for a greater percentage of the switch period. Therefore, the flow ratio will have a general tendency to increase with increases in the duty cycle. Take special note though that the rate of fluid momentum into the system, $\dot{\Gamma}_{in}$, is a function of $\alpha$ (refer to Equation 3.14). If one increases the duty cycle one should expect the inlet fluid momentum rate $\dot{\Gamma}_{in}$ to decrease, effectively reducing inlet flow $Q_{in}$. If less water flows into the pump, less water will flow out of it. This is evidenced by results shown in Figure 3.14. These results were garnered by iteratively simulating the nonlinear VSMS pump model for duty cycles ranging from 2% to 100%.

As suggested by the above analysis the flow ratio ($Q_{out}/Q_{in}$) does generally increase with $\alpha$. It does reach a maximum and begins to decrease suggesting a limit on how much the flow ratio can be increased. However, increasing the duty cycle has a negative effect on the the inlet flow $Q_{in}$ and therefore reduces the amount of water that can be transferred out of the pump. The results suggest that the flow out of the system can be maximized for a duty cycle near 10%. Additionally, the pressure at junction 2 $P_{jnc2}$ is also maximized at at the same duty ratio.

Using a sensor to track the motion of the waste-gate valve in the experimental setup (refer to §4.3 for further details), it was determined that the valve is at least partially open for about $\frac{1}{2}$ to $\frac{3}{4}$ of each cycle. The check valve is currently not monitored using a sensor but it is suspected that if the waste-gate valve is partially open for approximately $\frac{3}{4}$ of the cycle then the check valve is open for $\frac{1}{4}$ of the cycle. Currently, the
actuation of the valves is innate to the system; i.e., no outside controlled force opens and closes these valves. Hence, to adjust the pressure necessary to close each valve would require the the valves be redesigned or that an actuator be introduced. The designer of the Fleming pump exhausted much time and effort designing the waste-gate valve via trial and error. Increasing the pumping chamber capacitance is more feasible.

To increase the pump capacitance and improve pump performance, an air bladder was placed in the pumping chamber. The bladder was filled with air at a pressure slightly above atmospheric, and it occupied most of the chamber. The bladder is much
Figure 3.15: Experimental results and corresponding model predictions for pump chamber with air bladder in the pumping.

more compliant than water. Figure 3.15 shows experimental results and model predictions for the pump using the air bladder in the pumping chamber. For details about how the chamber is represented in the model when the air bladder is present refer to §A.1.

These results substantiate two important things. First, they further validate the pump model. Even though the pumping chamber representation has changed to accommodate the increased compliance (refer to §A.1), the model predictions match the magnitude and general trend of the lower-order dynamics of the experimental data. Second, the results also validate the suggested design improvement as evidenced by
### Table 3.1: Hydraulic-ram pump experimental data statistics.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>W/o Bladder</th>
<th>W/ Bladder</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{jnc2}$</td>
<td>6.33</td>
<td>8.80</td>
</tr>
<tr>
<td>$P_{jnc2}/P_{jnc1}$</td>
<td>13.01</td>
<td>13.81</td>
</tr>
</tbody>
</table>

As shown in the table, both the average junction 2 pressure $P_{jnc2}$ and average pressure boost $P_{jnc2}/P_{jnc1}$ are increased by introducing the air bladder. Though the values may not exactly match results suggested by Figure 3.13, they do follow the general trends predicted in that figure. Without the bladder the junction 2 pressure $P_{jnc2}$ reached a higher maximum but it varied over a greater ranged and dropped down near zero for a significant portion of the switch cycle (refer to Figure 3.12). With the bladder in place, the pressure varies less and stays at or near approximately 8.80 PSI. Much like the increase in capacitance led to lesser output voltage ripple, the increased compliance in the pumping chamber has led to decreased junction 2 pressure variation. The pressure remains closer to its mean value.

### 3.4 The Clutched Yo-Yo

Thus far we have looked at systems that employ switching elements to convert power. Let us turn our attention to the clutched yo-yo (refer to Figure 3.16). The clutched yo-yo employs a switch-like clutch that allows the yo-yo to free-wheel if and when the yo-yo
reaches an angular velocity fast enough to induce a centripetal force that overcomes the static spring force and disengages the brakes. When the yo-yo slows due to rotational damping the centripetal force decreases and the brake is re-engaged causing the yo-yo to retract. By controlling the initial velocity at which the yo-yo is released, one can control the amount of time the yo-yo free-wheels.

The yo-yo and its brake mechanism are an example of a centrifugal clutch. Centrifugal clutches have a myriad of general purpose uses including compressors, chippers/shredders, commercial lawn equipment, drilling equipment, construction equipment, amusement rides, go-karts, etc.\(^2\) The wide application of this type of mechanism

\(^2\)Refer to the Hilliard Corporation and Comet Industries web sites (www.hilliardcorp.com and...
makes the clutches yo-yo an ideal candidate to examine using VSMS. Lessons learned
from the yo-yo could potentially be more extensively applied to model commercial
centrifugal clutches in a wide variety of applications.

For the purpose of this study several simplifying assumptions are made, the
reasons of which will become more evident later. These assumptions are:

- The yo-yo does not slip relative to the string while ascending or descending. This
  assumption can be made because even if the brakes disengage during ascent or
descent the string provides enough friction along the inner walls of the yo-yo to
  prevent it from slipping.

- Slip only occurs during the free-wheeling stage when the yo-yo reaches the bot-
tom and the brakes disengage.

- The string tension will be assumed to apply a torque at a constant radius during
  ascent and descent.

- The brake masses move out radially at the same rate and therefore are always at
  the same radial position.

These assumptions allow one to separate the yo-yo dynamics into three sub-structures
– string tension right of center (Structure A), string tension left of center (Structure B),
and the free-wheeling yo-yo (Structure C) (refer to Figure 3.17).

www.hoffcocomet.com) for more applications.
Let us first assume that the yo-yo is dropped with no initial velocity. Carefully note the motion of the yo-yo. The yo-yo first descends with positive angular velocity (using the right-hand rule). Then reaches bottom and begins to ascend still rotating in a positive direction. Because of gravity the yo-yo begins to slow and comes to a stop. The yo-yo then begins to rotate in a negative direction until it reaches bottom and again ascends but still rotating in a negative direction. Take notice of how the string tension imparts a torque about the yo-yo’s center of mass. Upon initial descent, the string tension is right of center and imparts a positive torque about the center of mass (Structure A), but as the yo-yo ascends the string is then left of center and the tension imparts a negative torque (Structure B). The key here is whether the string torque is left or right of the center of mass and, therefore, whether it imparts a positive or negative torque.
torque. The torque is positive when the yo-yo is on the side of original descent and negative when it is on the side of original ascent; it does not depend specifically on whether it is ascending or descending.

The string, though, also accounts for a friction torque. As mentioned earlier, during ascent and descent, the string imparts a friction torque along the inner wall of the yo-yo and prevents the yo-yo from free-wheeling during ascent or descent. This dissipative torque depends on the direction of rotation – on whether $\dot{\theta}$ is positive or negative.

To fully understand the clutched yo-yo one must consider yet a third variable structure – the yo-yo free-wheeling (Structure C). This occurs when the brakes are disengaged and the yo-yo hits bottom. While free-wheeling, the yo-yo is not acted upon by a torque due to the string tension. There is, however, a damping torque that causes the yo-yo to slow when it is free-wheeling.

Three variable structures have been identified for the clutched yo-yo. To enhance the development of model for the clutched yo-yo, a simple yo-yo will be considered first.

### 3.4.1 Simple Yo-Yo Model Synthesis and Simulation

Both a simple yo-yo and a clutched yo-yo employ variable structures to accomplish a useful task. The simple yo-yo does not, however, employ a clutch to aid it in free-wheeling. If it is assumed that the string does not slip with respect to the yo-yo’s axle,
then only two structures remain due to whether the string tension imparts a positive or negative torque: Structure A and Structure B.

The system equations for a simple yo-yo can be determined using a variety of methods that include a kinematics approach or a Lagrange approach. Both of which provide the following system equations:

\[ J \ddot{\theta} = \alpha R_{\text{string}} T_{\text{string}} - \mu N \, \text{sgn}(\dot{\theta}) \]  
\[ (3.15) \]

and

\[ M \ddot{y} = T_{\text{string}} - M g \]  
\[ (3.16) \]

where

\[ \alpha = \begin{cases} 
-1 & \text{when } \theta < \frac{L_{\text{string}}}{R_{\text{string}}} \\
+1 & \text{when } \theta > \frac{L_{\text{string}}}{R_{\text{string}}} 
\end{cases} \]

and where \( M \) is the mass of the yo-yo and \( N \) a static torque equivalent to moment of the yo-yo weight about the radius \( R_s \) (i.e. \( N = M g R_s \)) and \( L_{\text{string}} \) is the unwound length of the string. Because of the no slip condition the sign of the torque can be determined directly from the measure of revolutions \( \theta \). The length of string traveled and hence the vertical position \( y \) can be directly related to the angular revolution \( \theta \). During original descent (i.e. \( T_{\text{string}} \) is right of center and imparts a positive torque (Structure A)) the
following relations hold true:

\[ R_{\text{string}} \theta = H - y, \]
\[ R_{\text{string}} \dot{\theta} = -\dot{y}, \text{ and} \]
\[ R_{\text{string}} \ddot{\theta} = -\ddot{y}. \]  

(3.17)

Here \( H \) is the height from which the yo-yo is dropped. The yo-yo reaches bottom once it has traveled the length of the string \( L_{\text{string}} \) and therefore travels through \( \theta = L_{\text{string}}/R_{\text{string}} \) revolutions to reach bottom. As it begins to ascend for the first time (i.e. \( T_{\text{string}} \) is left of center and imparts a negative torque (Structure B)) the relations change:

\[ R_{\text{string}} (\theta - R_{\text{string}} L_{\text{string}}) = y - (H - L_{\text{string}}), \]
\[ R_{\text{string}} \dot{\theta} = \dot{y}, \text{ and} \]
\[ R_{\text{string}} \ddot{\theta} = \ddot{y}. \]  

(3.18)

Now that the variable structures have been identified and the system equations determined, one can determine the stability of each substructure and map the system results over substructure results to view the switching surface. When the string imparts a positive torque (i.e. \( \alpha = +1 \)) the system operates like Structure A in Figure 3.17 (a), and the eigenvalues can be determined from Equation 3.15 as follows:

\[ J\ddot{\theta} = ( +1) R_{\text{string}} T_{\text{string}} - \mu N \text{sgn} (\dot{\theta}) \Rightarrow Js^2 = R_{\text{string}} T_{\text{string}} - \mu N \text{sgn} (\dot{\theta}) \]

where the characteristic roots are

\[ s_{1,2} = \pm \sqrt{[R_{\text{string}} T_{\text{string}} - \mu N \text{sgn} (\dot{\theta})]/J}. \]
The coefficient of friction is very small. Therefore the friction torque is much less
than the torque applied ($\mu N \ll R_{\text{string}} T_{\text{string}}$), and the characteristic roots are purely
real ($\text{Im} s_1 = \text{Im} s_2 = 0$) and of unlike signs ($s_2 < 0 < s_1$). These roots correspond
to an unstable system whose trajectories in the state plane are hyperbolas with two
asymptotes. It makes sense that in this mode the system is unstable. Imagine a yo-yo
with an infinitely long string; it would fall forever.

In a similar manner, Structure B stability can be examined. The characteristic
roots are determined using Equation 3.15 as follows:

$$J\ddot{\theta} = (-1)R_{\text{string}} T_{\text{string}} - \mu N \text{sgn}(\dot{\theta}) \Rightarrow J s^2 = -R_{\text{string}} T_{\text{string}} - \mu N \text{sgn}(\dot{\theta})$$

where the roots are

$$s_{1,2} = \pm \sqrt{-\left[R_{\text{string}} T_{\text{string}} + \mu N \text{sgn}(\dot{\theta})\right]}/J.$$

The roots are purely imaginary ($\text{Re} s_1 = \text{Re} s_2 = 0$) and have imaginary parts of unlike
signs ($\text{Im} s_2 < 0 < \text{Im} s_1$). The roots, therefore, correlate to ellipses in the state plane.
Structure B is a stable system that limit cycles. Like the results from Structure A, these
results also make sense. Imagine that the yo-yo could be released upward rather than
downward. The yo-yo, regardless if it had an infinitely long string, would return to its
initial state due to gravity.

Notice that the friction torque introduces a switch-like dynamic because of its
discontinuous behavior, but since the friction torque is much less than the string torque,
the characteristic curves in the state-plane only vary slightly from when angular velocity
$\dot{\theta}$ is negative to when it is positive. The curvature changes slightly as $\dot{\theta}$ goes from positive to negative and vice versa. Strictly speaking, there are therefore two switching surfaces – one due to angular position $\theta$ and the other to angular velocity $\dot{\theta}$.

![Simple yo-yo phase diagram](image)

Figure 3.18: Simple yo-yo phase diagram ($\dot{y}|_{t=0} = 0$).

The yo-yo modeled in this section has the following parameters:

- $R = 1\frac{3}{16}$ in = 3.02 cm
- $R_{string} = \frac{1}{4}$ in = 0.63 cm
- $L_{string} = 38$ in = 96.52 cm
- $M = 3.527$ oz = 100 g
- $H = 45$ in = 1.143 m
- $\mu = 0.1826$

The characteristic curves for Structure A and Structure B are mapped in Figure 3.18. The diagram is divided into quadrants. The quadrants represent the following scenarios:
Figure 3.19: Simple yo-yo results ($\dot{y}|_{t=0} = 0$).

I Yo-yo descending and string torque positive: $\alpha = +1$ and $\dot{\theta} > 0$.

II Yo-yo ascending and string torque negative: $\alpha = -1$ and $\dot{\theta} > 0$.

III Yo-yo descending and string torque negative: $\alpha = -1$ and $\dot{\theta} < 0$.

IV Yo-yo ascending and string torque positive: $\alpha = +1$ and $\dot{\theta} < 0$.

Mapped over these characteristic curves is the phase diagram for the simple yo-yo released from height $H$ with no initial velocity. The system switches between an unstable structure (Structure A) and an oscillatory structure (Structure B) to approach global asymptotic stability. As time approaches infinity ($t \to \infty$) the system approaches steady
state where \((\theta, \dot{\theta}) = \left(\frac{L_{\text{string}}}{R_{\text{string}}}, 0\right)\). The yo-yo comes to rest at the bottom end of the string.

Figure 3.19 shows the dynamic results for the yo-yo dropped from height \(H\) at zero velocity. The yo-yo climbs up a shorter height after every cycle. As the yo-yo reaches steady state, it comes to rest at the bottom of the string at a height \(H - L = 7\) inches. To reach bottom the yo-yo must revolve \(\frac{L_{\text{string}}}{R_{\text{string}}} \approx 24\) revolutions. Note that the dynamic results approach zero height and 24 revolutions near steady state.

### 3.4.2 Clutched Yo-Yo Model Synthesis and Simulation

The modeling of a clutched yo-yo presents some additional significant difficulties. Including the brake mechanisms in the model requires introducing more energy storing elements (refer to Figure 3.16). Furthermore, the varying radial position of the brake masses results in a variable rotational inertia. A third variable structure is introduced by the disengagement of the brakes which allows the yo-yo to function as a flywheel and rotate freely when it hits bottom.

The system equations for the clutched yo-yo require an additional equation. A third state is needed to track the radial position of the brake masses. The system equations for the clutched yo-yo are

\[
(J + 3mr^2) \ddot{\theta} + 6mr \dot{r} \dot{\theta} = \alpha_1 R_{\text{string}} T_{\text{string}} - (1 - \alpha_2) \mu N \text{sgn} (\dot{\theta}) - \alpha_2 B \dot{\theta},
\]

\[
M \ddot{y} = T_{\text{string}} - Mg,
\]
and

\[ \dot{m}r = mr\dot{\theta}^2 - k(r - r_0) \quad (3.21) \]

where

\[ \alpha_1 = \begin{cases} 
-1 & \text{when string is left of center of mass} \\
0 & \text{when the yo-yo free-wheels at the end of the string} \\
+1 & \text{when string is right of center of mass} 
\end{cases} \]

and where

\[ \alpha_2 = \begin{cases} 
0 & \text{when the yo-yo is descending or ascending, } y > H - L_{\text{string}} \\
1 & \text{when the yo-yo free-wheels at the end of the string, } y = H - L_{\text{string}} 
\end{cases} \]

In these equations \( m \) is the mass of each individual brake and \( M \) the mass of the complete yo-yo (brakes and all). Several other parameters have been introduced including the spring constant \( k \) and the radial position \( r_0 \), corresponding to the radial position of the spring’s free end when uncompressed. The numerical values of the additional parameters used for simulations are:

\[ M = 3.127 \text{ oz} = 89.5 \text{ g} \quad m = 0.0529 \text{ oz} = 1.5 \text{ g} \]

\[ k = 1.763 \text{ lb/in} = 308.7 \text{ N/m} \quad r_0 = \frac{7}{16} \text{ in} = 1.11 \text{ cm} \]

Because the friction torque is due to the string, it only applies when the yo-yo is descending or ascending, and not when it free-wheels. The system equations are nonlinear and determination of the stability of each variable structure requires nonlinear analysis. The stability for each cannot be as readily determined as in the previous section.
It can be more easily determined, though, how the structures behave about or near an equilibrium point by linearizing the system equations about that equilibrium point. If the equations are linearized about the origin using a small-signal approximation then the linearized equation are

$$J \frac{d^2 \hat{\theta}}{dt^2} = \alpha_1 R_{string} T_{string} - (1 - \alpha_2) \mu N \text{sgn} \left( \frac{d\hat{\theta}}{dt} \right) - \alpha_2 B \frac{d\hat{\theta}}{dt} , \quad (3.22)$$

$$M \frac{d^2 \hat{y}}{dt^2} = T_{string} - Mg , \quad (3.23)$$

and

$$m \frac{d^2 \hat{r}}{dt^2} = -k(\hat{r} - r_0) \quad (3.24)$$

where $\alpha_1$ and $\alpha_2$ are defined as above and where $\hat{\theta}, \hat{y}, \text{and} \hat{r}$ are the perturbations about the origin. The origin is chosen because it is an equilibrium point for both variable structures.

Note that the linearized equations are decoupled. Furthermore, Equations 3.22 and 3.23 are nearly the same as those used for the simple yo-yo except for the added damping term in Equation 3.22. Since the added term only applies when the yo-yo free-wheels, the linearized model suggests that the clutched yo-yo should function dynamically while descending and ascending like the simple yo-yo. The linearized model suggests that the characteristic curves for these structures should be similar to those mapped in the previous section. It seems reasonable that the clutched yo-yo should function similar to a simple yo-yo when ascending or descending. This leaves only the third structure to analyze.
When the yo-yo free-wheels it is at the end of the string and does not move vertically (i.e. $y = H - L$ and $\dot{y} = \ddot{y} = 0$). In addition, the damping torque applies, but the string and friction torque do not ($\alpha_1 = 0$ and $\alpha_2 = 1$). Therefore the characteristics for the equation of rotational motion are determined as follows:

$$J\ddot{\theta} = -B\dot{\theta} \Rightarrow Js^2 + Bs = s(Js + B) = 0$$

where the roots are

$$s_1 = 0 \text{ and } s_2 = \frac{B}{J}.$$  

This a degenerate case [11]. The phase trajectories are straight lines with a negative slope. There exists a line through the origin along which the motion is asymptotically stable.

By linearizing the system, the analysis developed for the simple yo-yo and the analysis detailed in the above paragraph for Structure C can be used to predict and better understand the dynamics of the clutched yo-yo. Figure 3.20 shows the phase diagram for the clutched yo-yo mapped over the linearized results for Structures A, B, and C. Though not exact, the clutched yo-yo results approximately follow the characteristic curves for Structure A when the yo-yo descends, Structure C when the yo-yo free-wheels, and Structure B when the yo-yo ascends. Note that the yo-yo abruptly comes to a stop because the initial velocity is fast enough to cause the yo-yo to return completely to the user’s hand. This is also shown by the simulation results in Figure 3.21.

Figure 3.21 contains the dynamic results for the yo-yo released from a height $H$.
Figure 3.20: Clutched yo-yo phase diagram ($\dot{y}|_{t=0} = -2 \text{ m/s}$).

with the initial velocities $\dot{y}|_{t=0} = 0$, $\dot{y}|_{t=0} = -39.4 \text{ in/sec} \approx -1 \text{ m/s}$, and $\dot{y}|_{t=0} = -78.7 \text{ in/sec} \approx -2 \text{ m/s}$. For the first two sets of results ($\dot{y}|_{t=0} = 0$ and $\dot{y}|_{t=0} = -39.4 \text{ in/sec}$) the clutched yo-yo responds just like the simple yo-yo. When released with zero initial vertical velocity the yo-yo falls and rises iteratively retracting less and less after every cycle just like the simple yo-yo. When released downward at an initial velocity of -39.4 in/sec the yo-yo has enough initial energy to fully retract to the user’s hand. The simulation is designed to stop if and when the yo-yo reaches maximum height after initial release. The third results show that the yo-yo free-wheels for approximately 1 second. Based on the results in Figure 3.21, the brakes will disengage and cause the
Figure 3.21: Clutched yo-yo dynamic results for several initial vertical velocities.

The detailed plot given in Figure 3.22 shows how the angular velocity changes as the yo-yo descends, free-wheels, and ascends. The brake is disengaged during the descent and free-wheel stages. The brake engages at an angular velocity of just under 45 rev/sec suggesting that for angular velocities greater than 45 rev/sec the brake will disengage and allow the yo-yo to free-wheel at the end of the string.
Figure 3.22: Detailed view of dynamic results for $\dot{y}|_{t=0} = -78.7 \text{ in/sec} \approx -2 \text{ m/s}$.

### 3.4.3 Yo-Yo Control

Now one can imagine that this model could be used to predict how long the yo-yo free-wheels for a given initial downward vertical velocity. Someone using a yo-yo might know that a given trick requires a specified amount of free-wheeling time. The yo-yo model is simulated iteratively to show how the free-wheeling time increases with initial downward velocity (refer to Figure 3.23). As expected, the faster one releases the yo-yo the longer it will rotate freely before retracting. A person is physically limited in how fast he/she can release a yo-yo and therefore the upper velocity range plotted in Figure 3.23 is probably beyond the physical ability of a person. Nonetheless, one could
Figure 3.23: Free-wheel time as a function initial downward velocity.

use these results to improve the timing of a trick or series of tricks.

3.5 Summary

It has been shown how variable structure system theory provides a formulation that can aid and guide in the model synthesis of switched systems like the boost converter, hydraulic-ram pump, and clutched yo-yo. Using Variable Structure Model Synthesis, one can reduce the complex switched system into more manageable continuous structures. The VSMS model aids the modeler in determining the key parameters that affect the system’s overall performance and the general trends associated with these param-
eters. The insight gained from these models suggested design improvements for both the boost converter and the pump. These models were validated by experimental results and in the case of the boost converter by comparison to a model derived from a more conventional method. This approach shows significant promise for synthesizing simplified models of switched systems.

In the next chapter, a metric will be presented for ascertaining the fidelity of a VSMS model. *Model fidelity* will be more clearly defined with specific emphasis to its application to VSMS models.
Chapter 4

Metrics for Measuring Model Fidelity

It is always important to verify and/or validate\(^1\) a model to assure that the prescribed objectives were sufficiently satisfied. For simple, more intuitive systems this may simply require observing simulation results and verifying that these results make engineering sense. For more complex systems, it is not always obvious from just observing simulation results that a model makes sense and that it effectively represents the actual system. The task of validating a model experimentally may be just as laborious as synthesizing that model.

The models in Chapter 3 have been confirmed in a number of ways. For instance, the boost converter simulation results were validated by experimental data that showed that the model is highly accurate. Additionally, design improvements suggested

\(^1\)Bennett [3] uses the term “verify” as ascertaining that a simulation behaves as expected. “Validate” is determining how closely the model emulates the real system.
by the VSMS model were also verified by simulation and validated by experiment results. The hydraulic-ram pump model was also validated experimentally in much the same way.

### 4.1 Definition of Model Fidelity

*Model Fidelity* is a measure of how precisely a model represents the physical behavior of a system. Fidelity is often quantified by comparing simulation results to experimental data or to a more complex model that precisely represents the physical system dynamics. Varying model fidelity can be approached in a top-down manner. One might have a complex, detailed model (i.e. a high fidelity model) and wish to systematically decrease the fidelity of that model to make the task of discerning key parameters and their general trends more tractable, as with the case of the hydraulic-ram pump. On the other hand a bottom-up approach may be more desirable; one may wish to start from scratch, with no model, and synthesize a model of desirable fidelity. One cannot however discuss fidelity of a model without providing substantive measures of that fidelity.

### 4.2 How to Measure Model Fidelity

The model fidelity must be tailored to meet the objectives prescribed in the problem statement. Model fidelity can be measured in a variety of manners. The methods used
depend on the application of the model. If, for instance, a prescribed objective is to synthesize a model that matches experimental results, one could use a calculation of the relative error to quantify the fidelity of the model and validate that the objective was met. In other cases, verification is sufficiently appropriate. For example, the desired pump model was the most simple model that would accurately identify key parameters and suggest in general terms how the pump’s performance was affected by these parameters. Before experimental results became available, it was ascertained whether the general trends predicted by the model made sense. Because the pump is a hydraulic analog of a boost converter, it was expected that increasing the compliance of the pumping chamber would decrease the junction 2 pressure variation. The model in Chapter 3 suggested so, and experimental results further validated this trend.

4.3 Quantifying Model Fidelity via Relative Error

As has been mentioned earlier, a simulation can be compared to experimental results to quantify its accuracy and access its validity. Every model outputs tracked states $y(t)$ that the user is interested in predicting. Sensors can be used experimentally to observe the actual dynamics of these states $y_a(t)$. A measure of the absolute estimation error is

$$e_{abs}(t) = |y_a(t) - y(t)|$$  \hspace{1cm} (4.1)

and its mean-squared value

$$\overline{(e_{abs}(t))^2} = \overline{y_a(t)^2} - 2\overline{y_a(t)y(t)} + \overline{y(t)^2}.$$  \hspace{1cm} (4.2)
The relative error is

\[ e_{\text{rel}}(t) = \left| \frac{y_a(t) - y(t)}{y_a(t)} \right| \]  

(4.3)

and its mean-squared value

\[ e_{\text{rel}}(t)^2 = \frac{e_{\text{abs}}(t)^2}{y_a(t)^2}. \]  

(4.4)

The relative error \( e_{\text{rel}}(t) \) can be used to quantify the fidelity of a model. If the user wishes to predict an observed state to within a specified percentage relative error, Equation 4.3 can be used to determine if over a specified time span the model prediction is within the specified relative error. If the model prediction exceeds the relative error, a higher fidelity model is required.

Figure 4.1: (a) Model error relative to experimental data. (b) Model error relative to higher fidelity model.
Imagine for example that one desires a model of the boost converter that predicts the output voltage to within 3\% relative error over the entire switch period. One can measure the fidelity of the state-space averaged and the VSMS models relative to the experimental data. The relative errors for the both models are plotted in Figure 4.2. If one only calculates the mean of the relative error over the switch period, both models seem reasonably accurate because the state-space averaged model has a mean relative error of 3.75\%, and the VSMS model has a mean relative error of 0.80\%. The results in Figure 4.2 suggest otherwise. Note that the for about half the switch period the state-space averaged model exceeds 3\% relative error. The relative error of the VSMS model
is less than 2% for most of the switch period. These results suggest two things: (1) the VSMS model is of higher model fidelity than the state-space averaged model and (2) the model is of a fidelity that meets the objective.

If experimental results are not available, a model can be compared to one of higher fidelity. This requires that a model of higher fidelity be available. This could be approached in a top down or bottom up manner. Imagine that experimental data is not available, but a detailed model that is deemed sufficiently accurate is available. Two possible scenarios are explored below:

1. **Scenario #1**: One wishes to synthesize a lower fidelity model from scratch that predicts system dynamics to within a specified relative error with respect to the higher fidelity model.

2. **Scenario #2**: One wishes to systematically simplify an existing model to one of lower fidelity that matches the original model to within a specified threshold.

Scenario #1 can be approached in much the same manner as the boost converter was handled in the previous section. A model can be synthesized and outputs of both the higher and lower fidelity models can be compared to calculate relative error and thus quantify the fidelity of the simpler model. The model can be iteratively augmented until the resulting model matches the higher fidelity model to within the specified relative error. Figure 4.3 shows an algorithm for using relative error to iteratively augment model fidelity and obtain the desired model. A similar procedure is used in nonlinear
analysis to produce spectral or Volterra models of nonlinear stochastic systems [2, 21]. The difficult issue is determining how to properly augment a model to include necessary elements to properly predict system dynamics.

Figure 4.3: Bottom-up approach for increasing model fidelity using relative error.

Relative error is only a “global” measure of the fidelity. It does not indicate if the model error is due to parameter uncertainty or an incorrect model structure. Furthermore, it does not help localize error due to specific model elements. By localizing the error, it is more manageable to determine where to augment fidelity in the model with more elements or more detailed constitutive relations. The following section explores a method for quantifying the relative importance of each model element.
4.4 Quantifying Model Fidelity via Element Activity

Scenario #2 requires that one identify the model elements that are less significant with respect to predicting the system dynamics. A model can be thought of as encompassing a set of inertial, compliant, and resistive elements (power storing or power dissipating elements) as is done in a bond graph approach. The key is then to determine how much each element contributes to the overall system dynamics. Once that is determined, elements can be systematically removed or added to a model to achieve a desired fidelity. For instance, the least important elements could be removed to simplify a model while maintaining sufficient accuracy.

Rosenberg and Zhou explored in [23] several power-based methods for quantifying the relative importance of model elements. They examined several measures including root-mean square power, average power, an average absolute power. Using these measures they implemented a tool in ENPORT-7 to visualize the relative importance of each model element [22]. Louca et al. expanded on this to develop the idea of element activity which they used to systematically reduce the order of a higher fidelity model. They used this idea in conjunction with model order reduction algorithms to produce reduced-order models of modal type problems.

This section explores how element activity can be used to (1) determine the relative contribution of each element to the prediction of the system dynamics, (2) reduce the order of a higher fidelity model, and (3) quantify the fidelity of a reduced
model relative to a higher fidelity model.

### 4.4.1 Element Activity

Louca et al. present in [15, 17] a power-based metric for quantifying the contribution of each power storing or power dissipating element (i.e. an $I$, $C$, or $R$) to the system dynamics and thus determine its *priority* in predicting the system behavior. (For a more thorough discussion of the following details refer to [17].) They introduce a means of measuring the energy flowing in and out of each element over a specified period of time. They apply this idea to already validated models. Introduced in [17] is the idea of element activity:

$$A = \int_0^\tau |P(t)| dt$$  \hspace{1cm} (4.5)

where $P(t)$ is the element power and $[0 \ \tau]$ is the time span over which the model will predict the physical system behavior. Note that activity $A$ has units of energy and that it represents the flow of energy through a specified element during time span $[0 \ \tau]$.

One can calculate the activity for each inertial, compliant, and resistive element assuming a constitutive law can be determined for each. The constitutive laws for inertial, compliant, and resistive elements can be represented in general terms using

\footnote{For purposes of this dissertation, the term “element” will be reserved solely for energy storing or dissipating elements and will not include sources, transformers, or gyrators.}
effort $e$ and flow $f$ variables as

\[ I: \quad f = \Phi_I(p) \iff p = \Phi_I^{-1}(f) \iff e = \dot{p} = \dot{f} \cdot \frac{\partial}{\partial f} \left( \Phi_I^{-1}(f) \right) \]

\[ C: \quad e = \Phi_C(q) \iff q = \Phi_C^{-1}(e) \iff f = \dot{q} = \dot{e} \cdot \frac{\partial}{\partial e} \left( \Phi_C^{-1}(e) \right) \]  

\[ R: \quad f = \Phi_R(e) \iff e = \Phi_R^{-1}(f) \]

where $\Phi_I$, $\Phi_C$, and $\Phi_R$ are known functions. With the constitutive laws of each element known, the powers needed for calculating each element activity are determined using the following:

\[ I: \quad P_I = e \cdot f = \dot{p} \cdot \Phi_I(p) = \dot{f} \cdot \frac{\partial}{\partial f} \left( \Phi_I^{-1}(f) \right) \cdot f \]

\[ C: \quad P_C = e \cdot f = \Phi_C(q) \cdot \dot{q} = e \cdot \dot{e} \cdot \frac{\partial}{\partial e} \left( \Phi_C^{-1}(e) \right) \]  

\[ R: \quad P_R = e \cdot f = e \cdot \Phi_R(e) = \Phi_R^{-1}(f) \cdot f \]  

To quantify the relative importance of each element it is necessary to compare the activity of each element to the overall activity:

\[ A_{\text{total}} = \sum_{i=1}^{k} A_i = \sum_{i=1}^{k} \left[ \int_{0}^{\tau} |P_i(t)| \, dt \right] \quad i = 1, \ldots, k \]  

where $A_i$ is the activity of the $i^{th}$ element of a total of $k$ elements. The activity index ($AI$) can be calculated for each element by normalizing each element activity by the total activity as follows:

\[ AI_i = \frac{A_i}{A_{\text{total}}} = \frac{\int_{0}^{\tau} |P_i(t)| \, dt}{\sum_{i=1}^{k} \left[ \int_{0}^{\tau} |P_i(t)| \, dt \right]} \quad i = 1, \ldots, k. \]  

In [15, 17], Louca et al. present in detail how activity analysis can be used along with a MORA to synthesize reduced-order models of linear, continuous-time systems. Though
this formulation generally holds for nonlinear systems, it has yet to be extensively applied to switched systems. It is important to note that application of activity analysis assumes a certain level of confidence in the model. If the model is ill implemented or synthesized the activity estimates are bogus (i.e. you must have a well verified model).

It is usually easier to derive or determine constitutive relations for linear, continuous-time systems. Nonlinear elements can introduce significant difficulties, but constitutive laws can many times be determined. Discontinuous systems have an added difficulty because constitutive relations abruptly change. By using a VSS formulation, however, it is possible to dissect a switched system into continuous generally nonlinear variable structures. The following shows how this idea can be more generally expanded to aid in the synthesis of nonlinear discontinuous switched systems.

4.4.2 Prioritizing Switched System Elements Via Activity Indices

The varying structure of the systems examined in Chapter 3 present an added difficulty that to date has not been evaluated using the element activity approach. Each variable structure is significantly unique, and model elements play varying roles while the system operates in each structure. In other words, a single element may have significantly different activity from one variable structure to the next. Each variable structure sub-model can be reduced in and of itself or the overall model can be simplified. It is therefore important to not only determine the overall activity of each element over time-span \([0 \, \tau]\) but rather to determine each element’s activity while the system operates in
In [15], Louca et al. show how the element activity approach is used to synthesize a reduced-order model of a vehicle suspension. The models used are linear, and the system input is the time derivative of the road profile – the road velocity. The element indices are determined for varying frequency sinusoidal road velocities. Louca et al. show that it is only necessary to calculate the activities over one cycle of the sinusoidal. The cyclic nature of the boost converter and hydraulic-ram pump facilitates the process of calculating element activity indices. Each is a power converter that operates at steady-state, so it is only necessary to calculate indices over one switch period.

The switch frequency at which the BJT is operated in the boost converter can be varied to affect output voltage ripple, but for a low-power converter the frequency is typically on the order of 1 kHz [10]. The activity indices must be calculated for various switch frequencies to determine if the importance of each element varies with the switch frequency. The converter modeled in Chapter 3 was operated with a duty cycle of 50%. The element activities \( A \) are numerically integrated during the converter simulation using Equation 4.5 and are output for post processing:

\[
A_I = \int_0^\tau \left( \frac{1}{L} \right) dt, \\
A_R = \int_0^\tau \left( \frac{R}{L} \right) dt, \\
A_C = \int_0^\tau \left( \frac{1}{C} \right) dt, \text{ and} \\
A_{R_{\text{load}}} = \int_0^\tau \left( \frac{1}{R_{\text{load}C}} \right) dt.
\]
The indices $AI$ are then calculated from these outputs for each half of the switch period $\tau_{sw}$ (i.e. $[0 \ \frac{1}{2}\tau_{sw}]$ and $[\frac{1}{2}\tau_{sw} \ \tau_{sw}]$) and then over the entire period $[0 \ \tau_{sw}]$ using Equation 4.9. Results are shown in Figure 4.4. Indices are determined for switch frequencies ranging from 100 Hz to 10 kHz using the model in Chapter 3.

![Figure 4.4: Activity indices for the electric boost converter.](image)

Interestingly, the indices calculated for the portion of the cycle that the diode conducts are more closely related to the indices averaged over the entire switch cycle. At lower frequencies the elements seem to have the same order of importance as for the entire cycle. Throughout the kHz range the load $R_{load}$, capacitance $C$, and inductance $L$ have nearly the same activity index and consequently equivalent importance.
The load $R_{load}$ is independent of the converter but should be considered when designing the converter. The inductor resistance $R$ plays the least important role throughout the frequency range examined. The results suggest that at lower frequencies the two more important model elements are the inductance $L$ and capacitance $C$. At lower frequencies, when the BJT conducts the inductor dominates, and when the diode conducts the capacitor dominates. Additionally, at lower frequencies, the capacitor is of higher activity index than the inductor over the entire cycle supporting what was suggested in the previous chapter – that the first place to begin to affect converter dynamics is by adjusting capacitance.

More importantly, though, results suggest that at higher frequencies the model could be reduced by eliminating $R$ while maintaining over a 95% threshold. Because the model was previously validated in Chapter 3 and shown to be accurate, the reduced model should still predict the converters dynamics relatively accurately. In other words, the reduced model would still maintain a high level of fidelity.

The hydraulic-ram pump can be examined in much the same manner that the converter was done except that unlike the converter the pump is not operated over a range of switch frequencies. The frequency at which the waste-gate valve is actuated is inherent to the pump (specifically the waste-gate valve design) as mentioned in Chapter 3. Therefore, it is only necessary to calculate indices at a single switch frequency. The
Table 4.1: Hydraulic-ram pump activity indices for the complete cycle and for the portions of the cycle that the waste-gate valve (WGV) closed and opened.

The indices are calculated for the time span that the waste-gate valve is closed $[0 \cdot D \cdot \tau_{sw}]$, the time span that the valve is open $[D \cdot \tau_{sw} \cdot \tau_{sw}]$, and for the complete cycle $[0 \tau_{sw}]$ (where $D$ is the duty cycle at which the waste-gate valve opens and closes). The indices are plotted in tabulated in Table 4.1 and Figure 4.5. The two most important elements are indicated in boldface in Table 4.1. The inlet hose fluid inertia $I_{inlet}$ and the pumping
chamber compliance $C_{pc}$ dominate the energy flow throughout the cycle. Throughout the cycle the inlet pipe fluid resistance has the lowest activity index.

Figure 4.5: Activity indices and dynamics for hydraulic-ram pump

The clutched yo-yo is a bit more difficult to examine than the pump and the converter because unlike those two systems the yo-yo does not operate in a limit cycle at steady-state. It is necessary to then calculate indices for the yo-yo from the moment is released from the user’s hand to the moment it retracts back the the hand and comes to a stop. Specifically, indices must be calculated for the time span that the yo-yo
Table 4.2: Clutched yo-yo activity indices for an initial velocity of $v|_{t=0} = -78.7 \text{ in/s}$ ($\approx -2 \text{ m/s}$).

<table>
<thead>
<tr>
<th>ELEMENT</th>
<th>DESCENT</th>
<th>FREE-WHEEL</th>
<th>ASCENT</th>
<th>COMPLETE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{yo-\text{yo rotation}} = I_1$</td>
<td>0.389</td>
<td>0.415</td>
<td>0.769</td>
<td>0.528</td>
</tr>
<tr>
<td>$I_{yo-\text{yo vertical translation}} = I_2$</td>
<td>0.036</td>
<td>0.000</td>
<td>0.072</td>
<td>0.037</td>
</tr>
<tr>
<td>$I_{\text{brake radial translation}} = I_3$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$I_{\text{brake rotation}} = I_4$</td>
<td><strong>0.358</strong></td>
<td>0.072</td>
<td>0.027</td>
<td><strong>0.276</strong></td>
</tr>
<tr>
<td>$C_{\text{spring}} = C$</td>
<td>0.186</td>
<td>0.028</td>
<td>0.000</td>
<td>0.138</td>
</tr>
<tr>
<td>$R_{\text{stick-slip friction}} = R_1$</td>
<td>0.031</td>
<td>0.000</td>
<td><strong>0.132</strong></td>
<td>0.021</td>
</tr>
<tr>
<td>$R_{\text{damping}} = R_2$</td>
<td>0.000</td>
<td><strong>0.485</strong></td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The above activities are derived in detail in Appendix B. Table 4.2 and Figure 4.6 show the indices for the clutched yo-yo when it is released downward at a rate of $v|_{t=0} =$
−78.7 in/s (≈ 2 m/s).

Figure 4.6: Activity indices for the clutched yo-yo released from an initial velocity of $v_{t=0} = -78.7 \text{ in/s} (\approx 2 \text{ m/s})$.

As the yo-yo descends the rotational inertia of the yo-yo and the brake masses dominate the energy flow. The damping only takes affect when the yo-yo free-wheels therefore the activity associated with this element is zero. The brake’s radial inertia activity index is not exactly zero but is much smaller than the rest and thus relatively negligible. As the yo-yo free-wheels the viscous damping and yo-yo rotational inertia are of highest priority. The stick-slip friction due to the string applies only when the yo-
yo ascends and descends and so the stick-slip friction does not have activity when the
yo-yo free-wheels. Also the yo-yo does not move vertically as it free-wheels making
its translational inertia activity zero. During ascension, the rotational inertia dominates
followed far behind by stick-slip friction. The activity index for the spring compliance
is zero because the brakes are engaged and spring force remains constant as the yo-yo
ascends. The damping no longer applies and therefore its index is zero.

When evaluating indices for the complete cycle an important consequence be-
comes evident. The damping activity is zero while the yo-yo ascends and descends. If
its activity is only calculated over the complete time span $[0 \tau_{final}]$ using Equation 4.9
it would appear to lend nothing to the yo-yo’s dynamics. ($\tau_{final}$ is the moment that the
yo-yo retracts back the the user’s hand and comes to a stop.) It has been shown, though,
that it plays an important role when the yo-yo free-wheels. This serves as an example
to prove that for a switched system it is not sufficient to calculate the activity indices
over the entire time span or even just one cycle; it is necessary to do so for each interval
that each variable structure applies. Each variable structure model can be individually
simplified to get a sufficiently accurate model as will be shown in the next section.

4.4.3 The Use of Activity Indices to Refine Model Fidelity

As shown in the previous section, activity analysis can be used to determine the rela-
tive importance of each element in a model. Model elements of negligible priority can
be eliminated from the model or variable structure sub-model to simplify the model.
Louca et al. in [15] present a “bottom-up” algorithm for synthesizing a “proper” model using activity indices. By bottom-up it is meant that the model is built up from scratch. Starting with the most important element, elements are iteratively augmented to the model. Based on engineering specifications, a threshold of the total activity (e.g. 95%) can be specified. Using activity indices, elements can be sorted by priority from highest to lowest and numbered $1, \ldots, k$. Then, starting with element 1 and working up,
elements can be iteratively augmented and a cumulative index $CI$ calculated each step until the threshold is exceeded. For example, after the $i^{\text{th}}$ iteration the cumulative index $CI$ would be

$$CI = \sum_{j=1}^{i} A_{j}$$

where $i \leq k$.

Once the threshold is exceeded, the remaining elements are rejected. The proper model consists of those elements incorporated by means of the above iterative process.

It may be easier though to use a “top-down” approach because the elements to be eliminated are likely far fewer than those to be included in the model. Starting with element $k$ and working down, elements are iteratively eliminated and the cumulative index calculated. This process continues until the resulting model has a cumulative index less than threshold. It is then necessary to step back one element and include it in the model so that the resulting cumulative index equals or just exceeds threshold. The remaining elements are included in the proper model. Both processes are detailed as an algorithm in Figure 4.7.

As mentioned earlier, because of the varying structure of switched systems the complete model or the variable structure models can be simplified. It is necessary then to calculate indices and prioritize elements for each variable structure. Take for example the boost converter model. Using a bottom-up approach, the model elements are prioritized as detailed in Table 4.3. The cumulative index can be calculated for each variable structure as each element is augmented in order of priority as also shown in Table 4.3. If, for instance, the original problem statement suggests that a 95% total
Table 4.3: Boost converter element ranks, activity indices $AI$, and cumulative indices $CI$ at a switch frequency of 1 kHz.

<table>
<thead>
<tr>
<th>RANK</th>
<th>ELEMENT</th>
<th>DIODE CONDUCTING</th>
<th>BJT CONDUCTING</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Element</td>
<td>$AI$</td>
<td>$CI$</td>
</tr>
<tr>
<td>1</td>
<td>$I$</td>
<td>0.3258</td>
<td>0.3258</td>
</tr>
<tr>
<td>2</td>
<td>$C$</td>
<td>0.3246</td>
<td>0.6504</td>
</tr>
<tr>
<td>3</td>
<td>$R_{load}$</td>
<td>0.3225</td>
<td>0.9729</td>
</tr>
<tr>
<td>4</td>
<td>$R$</td>
<td>0.0271</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

The activity threshold is necessary to meet prescribed model requirements, the inductor resistance (or resistor) $R$ can be eliminated from the model.

Figure 4.8: Boost converter reduced model.

The reduced model is shown in Figure 4.8. The resulting model system equations are

$$\dot{\lambda} = V_{in} - 0 - \alpha \left[ \frac{1}{C} q \right] \quad \text{and} \quad \dot{q} = \alpha \left[ \frac{1}{L} \lambda \right] - \frac{1}{R_{load} C} q.$$
The reduced model matches the original well at steady-state. Relative error can be calculated using Equation 4.3. Results are shown in Figure 4.9. The resulting model has an average relative error of 8% for both the prediction of inductor current $i_L$ and capacitor voltage $v_C$. The reduced model matches the general trend of the dynamics of the original model, plus it also predicts a similar amount of output voltage ripple. Though this model would have a higher error relative to the experimental data, it is still well suited for use in chapter 3 in §3.2 in place of the model used there to suggest redesign and predict converter dynamics.

Figure 4.9: Relative error for reduced boost converter model.

In much the same manner the hydraulic-ram pump model could be simplified.
It has already been stated that the purpose of the model is to predict general trends and suggest redesign. A high fidelity model would be intractable and ill suited for such a task. The lowest fidelity model that properly predicts general trends would be more desirable. It was shown in Chapter 3 that the current model does appropriately predict the general trends and lower-frequency dynamics of the pump. If the current model can be further simplified while still sufficiently predicting general trends, it would be desirable to use such a model because it would facilitate identification of key parameters and general trends. Activity analysis can be used to determine if the current model can be further simplified.

<table>
<thead>
<tr>
<th>RANK</th>
<th>ELEMENT</th>
<th>WGV CLOSED</th>
<th>WGV OPEN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Element</td>
<td>CI</td>
<td>Element</td>
</tr>
<tr>
<td>1</td>
<td>$C_{pc}$</td>
<td>0.463</td>
<td>$I_{inlet}$</td>
</tr>
<tr>
<td>2</td>
<td>$I_{inlet}$</td>
<td>0.852</td>
<td>$C_{pc}$</td>
</tr>
<tr>
<td>3</td>
<td>$I_{outlet}$</td>
<td>0.936</td>
<td>$R_{outlet}$</td>
</tr>
<tr>
<td>4</td>
<td>$R_{outlet}$</td>
<td>0.979</td>
<td>$I_{outlet}$</td>
</tr>
<tr>
<td>5</td>
<td>$R_{inlet}$</td>
<td>1.000</td>
<td>$R_{inlet}$</td>
</tr>
</tbody>
</table>

Table 4.4: Hydraulic-ram pump element ranks and cumulative indices $CI$.

In Table 4.4, the pump elements are ranked by priority and the cumulative indices are listed for the variable structures of the pump model. The results tabulated show that the inlet pipe resistance $R_{inlet}$ could be neglected while still maintaining over
95% total activity threshold for both structures (WGV CLOSED and WGV OPEN).

Physically, this is likely due to the inlet pipe dimensions and relatively smooth surface. The pipe has a significantly greater cross-sectional area. As is noted in §A.2 the pressure drop due to friction in the pipe is inversely proportional to the square of the cross-sectional area $A^2$. If the inlet pipe resistance is neglected as shown in Figure 4.10 the reduced model system equations become

\[
\dot{\Gamma}_{in} = P_{in} - 0 - \alpha \left[ \frac{1}{C_{pc}} V_{pc} \right],
\]
\[
V_{pc} = \alpha \left[ \frac{1}{I_{inlet}} \Gamma_{in} \right] - \frac{1}{I_{outlet}} \Gamma_{out}, \text{ and}
\]
\[
\dot{\Gamma}_{out} = \frac{1}{C_{pc}} V_{pc} - \frac{R_{outlet} (\Gamma_{out})}{I_{outlet}} \Gamma_{out} - P_{out}.
\]

Figure 4.10: Hydraulic-ram pump reduced model.

Figure 4.11 shows simulation results for both the original model and the reduced model in which inlet pipe friction $R_{inlet}$ is neglected. The reduced model closely follows the original model. Maximum relative errors appear to occur where the curves reach their maximum. At their maximum the relative errors are 9.94%, 8.81%, 10.91%, and
10.91% for $Q_{in}$, $Q_{out}$, $P_{jnc1}$, and $P_{jnc2}$ respectively. As has been shown in Appendix A, the friction loss in the inlet pipe is modeled by a nonlinear constitutive law. By eliminating this element the model can be significantly simplified while maintaining less than 11% relative error on all model output predictions. Furthermore, the reduce model is still well suited for predicting general trends and suggesting design improvements.

![Graphs showing volumetric flowrates and junction pressures](image)

Figure 4.11: Reduced model results for hydraulic-ram pump.

Thus far, energy elements have been eliminated solely based on their activity. As the following application will show, this approach cannot always be blindly applied without using some “engineering common sense.”

The clutched yo-yo has three separate variable structures; elements must be
ranked and cumulative indices $CI$ calculated for each structure. As evidenced by the model derivation in Appendix B, the yo-yo model is significantly more complex than the converter model. It would be desirable to reduce the model to aid the user in deriving intuitive conclusions about the yo-yo’s dynamics.

<table>
<thead>
<tr>
<th>RANK</th>
<th>ELEMENT</th>
<th>DESCENT</th>
<th>Free-Wheel</th>
<th>ASCENT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Element</td>
<td>$CI$</td>
<td>Element</td>
<td>$CI$</td>
</tr>
<tr>
<td>1</td>
<td>$I_1$</td>
<td>0.389</td>
<td>$R_2$</td>
<td>0.485</td>
</tr>
<tr>
<td>2</td>
<td>$I_4$</td>
<td>0.747</td>
<td>$I_1$</td>
<td>0.900</td>
</tr>
<tr>
<td>3</td>
<td>$C$</td>
<td>0.933</td>
<td>$I_4$</td>
<td>0.972</td>
</tr>
<tr>
<td>4</td>
<td>$I_2$</td>
<td>0.969</td>
<td>$C$</td>
<td>1.000</td>
</tr>
<tr>
<td>5</td>
<td>$R_1$</td>
<td>1.000</td>
<td>$I_3$</td>
<td>1.000</td>
</tr>
<tr>
<td>6</td>
<td>$I_3$</td>
<td>1.000</td>
<td>N/A</td>
<td>——</td>
</tr>
</tbody>
</table>

Table 4.5: Clutched yo-yo element ranks and cumulative indices $CI$ when released at $y|_{t=0} = -78.7$ in/s ($\approx 2$ m/s).

Table 4.5 gives the element ranks and cumulative indices for each variable structure of the yo-yo model. As discussed earlier, when the yo-yo descends, only 6 elements are active because the 7th element $R_{damping}$ does not apply when the yo-yo ascends or descends. As the yo-yo free-wheels, it does not move vertically, and therefore its translational inertia does not apply resulting in only 5 active elements. There are only 5 elements active when the yo-yo ascends because the brakes are re-engaged and the
spring does not move. For all three structures the least active element is the brake radial inertia $I_3$. Its activity index is always negligible (i.e. $< 0.001$). This element could be removed while still maintaining over a 99% total activity threshold.

Figure 4.12: Reduced model results for clutched yo-yo released at $y|_{\tau=0} = -78.7 \text{ in/s}$ ($\approx 2 \text{ m/s}$).

Figure 4.12 shows reduced model results for the clutched yo-yo. These results are achieved by neglecting the radial brake inertia. The system equations become

\[
(J + 3m_{\text{brake}}r^2)\ddot{\theta} + 6m_{\text{brake}}rr\dot{\theta} = \alpha_1 R_{\text{string}} T_{\text{string}} - (1 - \alpha_2)\mu N \text{sgn}(\dot{\theta}) - \alpha_2 B \dot{\theta},
\]

\[
MY = T_{\text{string}} - Mg, \text{ and}
\]

\[
\theta = m_{\text{brake}}r\dot{\theta}^2 - k(r - r_0).
\]  

(4.10)
The third equation is no longer differential but rather an algebraic constraint. The reduced results match the original model very closely.

Closer examination of Table 4.5 reveals that the individual variable structure sub-models could be further simplified while still maintaining about a 97% total activity threshold. For instance, the stick-slip friction $R_1$, spring compliance $C$, and brake rotational inertia $I_4$ could be neglected in the DESCENT, FREE-WHEEL, in the ASCENT sub-models respectively.

Care must be taken though, and some engineering sense used. If the spring compliance element is eliminated in the FREE-WHEEL sub-model then there no longer exists a restoring force that re-engages the brakes and causes the yo-yo to retract. This points to a significant issue. The yo-yo has a sub-structure that individually undergoes its own mode transitions – the clutch. The clutch is a sub-system of the yo-yo. The brakes are disengaged during both DESCENT and FREE-WHEEL and engaged during ASCENT. If the activity indices are calculated over both DESCENT and FREE-WHEEL while the brakes are DISENGAGED, the spring has a 0.164 activity index (recorded in Table 4.6) requiring that it be retained. The brake radial inertia $I_3$ remains a negligible element while the brakes are DISENGAGED.

By splitting the simulation into three sub-models (for DESCENT, FREE-WHEEL, and ASCENT) the model is further reduced. During DESCENT, in addition to the brake
Table 4.6: Element ranks and cumulative indices $CI$ calculated for time spans that brakes are ENGAGED (DESCENT and FREE-WHEEL) and DISENGAGED (ASCENT).

While FREE-WHEELing, the spring element must be retained and so the sub-model remains the same as the FREE-WHEEL sub-model used in the previous reduction (Equation 4.10). The rotational inertia of the brake is additionally neglected for the ASCENT.
sub-model. Therefore, the sub-model is

\[(J + 0)\dot{\theta} + 0 = \alpha_1 R_{string} T_{string} - (1 - \alpha_2)\mu N \text{sgn}(\dot{\theta}) - \alpha_2 B \dot{\theta},\]

\[M \ddot{y} = T_{string} - Mg,\]

and

\[0 = m_{brake} r \dot{\theta}^2 - k(r - r_0).\]

![Graphs showing angular position, brake position, and vertical position over time.](image)

Figure 4.13: Clutched yo-yo results after further reduction.

Results of this further reduction are plotted if Figure 4.13. These results still match the general trend of the original model quite well, and they show that it is possible to split a model into sub-models that represent the various modes of the system. This example shows that, to appropriately reduce a model, care must be taken in determining the proper time spans over which to calculate activity indices.
4.5 Iterative Model Synthesis

What if neither experimental results nor a high fidelity model are available to compare a simulation to? How would one quantify the fidelity of that model? This can be approached in similar manner to variable-step numerical integration routines where the step is iteratively decreased until the state-trajectory predictions for successive iterations negligibly differ. If the fidelity of a model is iteratively augmented successive models can be compared until the relative difference is negligible. Activity thresholds and relative errors can be used to iteratively synthesize a model from scratch and measure the relative difference between successive models.

This approach was used to develop the algorithm in Figure 4.14. The basic idea of the algorithm is to iteratively augment elements until any additional elements only make marginal improvements in predicting system dynamics. This algorithm requires 4 inputs: the total activity threshold, the element activity threshold, the relative error tolerance, and the initial model. Elements are ranked from highest activity index to lowest and numbered 1,\ldots, n. Then cumulative index is calculated for elements 1,\ldots, n−1:

\[
C_{I_i} = \sum_{i=1}^{n-1} A_{I_i}.
\]

The element activity threshold is used to determine if the \( n^{th} \) element is of sufficient importance to keep in the model. The relative error tolerance is used to determine if the relative error between outputs of successive models (i.e. a model that includes elements 1,\ldots, n−1 and another that includes elements 1,\ldots, n) is negligible. Two tests are used
Figure 4.14: Iterative algorithm for synthesizing model using activity analysis.

to confirm convergence between successive model outputs. The algorithm checks for convergence between successive models first by using the cumulative index and second by means of the relative error. The necessity for this seemingly redundant check will become more evident in the example that follows. For now, it is there to guarantee that the last element augmented is not so insignificant so as to meet the cumulative index requirement while not significantly improving the model output predictions.

The above algorithm is useful for problems like the hydraulic-ram pump. As-
sume for the sake of this exercise that experimental data for the pump is not available. The pump differs from the converter and yo-yo because it is not as straight-forward to parameterize as the other two. The boost converter and yo-yo have immediately identifiable physical elements with parameters that can be measured and quantified with readily available tools like a scale, a ruler, a multimeter, etc. The pump parameters are not as easily measured. As has been shown in Appendix A, the constitutive relations for the pipe and hose losses are nonlinear and change with respective flowrates. How does one know that they have synthesized a model that sufficiently predicts outputs so as to achieve the prescribed model requirements?

At first glance it may not be apparent how many elements are necessary to sufficiently predict the pump dynamics. A minimum order model must have at least enough elements to predict the desired outputs (i.e. $Q_{in}$, $Q_{out}$, $P_{jnc1}$, and $P_{jnc2}$). Three elements are necessary to predict these outputs – the inlet pipe fluid inertia $I_{inlet}$, the pumping chamber compliance $C_{pc}$, and the outlet hose fluid inertia $I_{outlet}$. The INITIAL MODEL is shown in Figure 4.15 (a).

Elements were iteratively augmented to the INITIAL MODEL using the algorithm in Figure 4.14 and results are documented in Tables 4.7 and 4.8. Simulation results for each model are plotted in Figure 4.16. The total activity threshold, element activity threshold, and relative error tolerance used to iteratively synthesize the pump model are 95%, 5%, and 15% respectively.

Before, augmenting an element it is necessary to check the INITIAL MODEL.
This is accomplished by calculating the cumulative index for elements 1, \ldots, n - 1. As shown in Table 4.7 the cumulative index for the Initial Model is 0.732. The first check is negative (i.e. $CI^* < 0.95$) and the $n^{th}$ element, $I_{\text{outlet}}$, has activity index of 0.268 which is greater than the element activity threshold. (Cumulative index for elements 1, \ldots, n - 1, $CI^*$, is in boldface.) All elements are retained and the next element is augmented.

The next element added is the inlet pipe fluid resistance $R_{\text{inlet}}$. Up till now the pipe and hose losses have been neglected. There is greater flow through the pipe, so considering this, the pipe loss might be a good place to start. The first test is passed because $CI^* = 0.965$). The algorithm precedes to the second test to see if the relative errors are within tolerance. The outputs are compared between the Initial Model.
<table>
<thead>
<tr>
<th>Rank</th>
<th>Element</th>
<th>$AI$</th>
<th>$CI$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_{pc}$</td>
<td>0.433</td>
<td>0.433</td>
</tr>
<tr>
<td>2</td>
<td>$I_{inlet}$</td>
<td>0.299</td>
<td>0.732</td>
</tr>
<tr>
<td>3</td>
<td>$I_{outlet}$</td>
<td>0.268</td>
<td>1.000</td>
</tr>
</tbody>
</table>

**INITIAL MODEL**

<table>
<thead>
<tr>
<th>Rank</th>
<th>Element</th>
<th>$AI$</th>
<th>$CI$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_{pc}$</td>
<td>0.413</td>
<td>0.413</td>
</tr>
<tr>
<td>2</td>
<td>$I_{inlet}$</td>
<td>0.302</td>
<td>0.715</td>
</tr>
<tr>
<td>3</td>
<td>$I_{outlet}$</td>
<td>0.250</td>
<td>0.965</td>
</tr>
<tr>
<td>4</td>
<td>$R_{inlet}$</td>
<td>0.035</td>
<td>1.000</td>
</tr>
</tbody>
</table>

**SECOND MODEL**

<table>
<thead>
<tr>
<th>Output</th>
<th>$e_{rel}$</th>
<th>Within Tolerance?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{in}$</td>
<td>0.237</td>
<td>No</td>
</tr>
<tr>
<td>$Q_{out}$</td>
<td>0.305</td>
<td>No</td>
</tr>
<tr>
<td>$P_{jnc1}$</td>
<td>0.490</td>
<td>No</td>
</tr>
<tr>
<td>$P_{jnc2}$</td>
<td>0.959</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 4.7: Hydraulic-ram pump element activity indices, cumulative indices, and mean relative errors for *INITIAL* and *SECOND MODELS*.

that does not include element $R_{inlet}$ and the *SECOND MODEL* that does. This model fails the second test. The algorithm then checks the activity index of the $n^{th}$ element (in this case element $R_{inlet}$). This element does not meet the element activity threshold because its index is less than 5%; it is removed from the model and another element is augmented. This element can be used later.

The outlet hose fluid resistance is then augmented to the model. The inlet pipe losses proved to be relatively negligible so maybe the outlet hose losses are more important. The resulting model (the *THIRD MODEL* in Figure 4.15) shows definite improve-
<table>
<thead>
<tr>
<th>Rank</th>
<th>Element</th>
<th>$AI$</th>
<th>$CI$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>THIRD MODEL</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$C_{pc}$</td>
<td>0.373</td>
<td>0.373</td>
</tr>
<tr>
<td>2</td>
<td>$I_{inlet}$</td>
<td>0.350</td>
<td>0.723</td>
</tr>
<tr>
<td>3</td>
<td>$R_{outlet}$</td>
<td>0.160</td>
<td><strong>0.883</strong></td>
</tr>
<tr>
<td>4</td>
<td>$I_{outlet}$</td>
<td>0.117</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td><strong>FINAL MODEL</strong></td>
<td>Output</td>
<td>$\epsilon_{rel}$</td>
</tr>
<tr>
<td>1</td>
<td>$C_{pc}$</td>
<td>0.360</td>
<td>0.360</td>
</tr>
<tr>
<td>2</td>
<td>$I_{inlet}$</td>
<td>0.343</td>
<td>0.703</td>
</tr>
<tr>
<td>3</td>
<td>$R_{outlet}$</td>
<td>0.149</td>
<td>0.852</td>
</tr>
<tr>
<td>4</td>
<td>$I_{outlet}$</td>
<td>0.116</td>
<td><strong>0.968</strong></td>
</tr>
<tr>
<td>5</td>
<td>$R_{inlet}$</td>
<td>0.032</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 4.8: Hydraulic-ram pump element activity indices, cumulative indices, and mean relative errors for **THIRD** and **FINAL MODELS**.

ment over the **INITIAL MODEL**. As shown in Table ??, it has a cumulative index for elements $1, \ldots, n-1$ that is greater than that for the **INITIAL MODEL** ($CI^* = 0.883$). The plotted results look good (refer to Figure 4.16) and match what one might intuitively expect. However, the cumulative index $CI^*$ does not exceed the total activity threshold specified. The $n^{th}$ element, $I_{outlet}$, is relatively important, and because its index exceeds the element activity threshold it is retained.
The inlet pipe resistance $R_{inlet}$ is again augmented in the Final Model. The first test is passed (i.e. $CI^* > 95\%$). The algorithm goes on to the second test. The relative errors for each output are less than tolerance as evidenced by Table 4.8. Both tests are passed. The desired model has been derived, and it is the same as the model used in Chapter 3 for the pump.

![Volumetric Flowrates and Junction Pressures](image)

Figure 4.16: Iterative results for hydraulic-ram pump using algorithm in Figure 4.14.

Figure 4.16 shows how each iteration marks definite improvement. It also shows how the model outputs computed converge more and more with each iteration. The last two models correlate well with each other relative to the first two. Either of the final two models could be used, but if no experimental data is available it would be prudent
to use the higher fidelity model – the FINAL MODEL.

4.6 Summary

Several metrics have been presented for quantifying the fidelity of a model. Methods have been developed for doing so whether experimental data is available, a higher fidelity model is available, or if neither are available. An iterative algorithm can be used to augment a model to achieve a desired fidelity. If experimental data is available the algorithm in Figure 4.3 can be applied to increase model fidelity. Activity indices provide a metric for determining the relative importance of each model element. It has been shown how activity analysis can be extended for use in switched model synthesis. This approach shows promise for synthesizing reduced models from higher fidelity models. A new algorithm as shown in Figure 4.7 has been presented for synthesizing models from scratch. This iterative approach uses activity analysis and relative error calculations to quantify the relative fidelity of each model synthesized.

These metrics and methods provide systematic ways for synthesizing a model of desired fidelity. The metric used is highly dependent on the problem statement, objectives, and the information available to the modeler.
Chapter 5

Extended Applications

The methods and metrics presented in the previous chapters provide a systematic approach for examining the test cases. They are, however, not limited to these cases alone. While developing the material presented thus far, several interesting ideas came to fruition. The first of which is to use the iterative model algorithm presented in Chapter 4 for a more general class of problems. For instance the iterative process could be used to synthesize a vehicle model. A second idea has been previously expressed in Chapter 1 and is restated here:

“Complex systems can be thought of in the framework of variable structures where each structure is a sub-model that emphasizes the system’s dynamics in a specific mode, time span, or frequency range...”
This idea was specifically motivated by the modeling of the clutched yo-yo. The second reduced yo-yo model is made up of three variable structure, sub-models each of which emphasize different dynamics during descent, free-wheeling, and ascent respectively. It has been shown how activity analysis can be used to reduce the overall model or alternatively each variable structure, sub-model.

Auto manufacturers are increasingly moving towards virtual product development. A vehicle includes many subsystems including suspension, electronics, and drive-train to name a few. Each has its own physical dynamics but all must interact as one. A complete model that includes all subsystems has innumerable parameters and can be cumbersome to understand. It would be more desirable that the model be tailored to serve the specific tasks of each engineering group. For example the suspension design team would prefer a model that only includes the pertinent information about the suspension dynamics. This can be approached in two ways as suggested in Chapter 4. A model can be synthesized from scratch to include only those elements of interest. Yet another approach is to simplify an existing model.

This brings up the idea of variable fidelity models – models whose fidelity is varied to capture only the pertinent information necessary to meet problem statement objectives. This can be taken a step further. A model’s fidelity can be altered to capture the systems dynamics for different time spans over which the system operates in variable modes. This will become more clear in the examples that follow.
5.1 Iterative Vehicle Model Synthesis

Iterative model synthesis is not limited to switched systems. It is applicable to a more general case of problems. The main advantages of using an algorithmic approach like that presented in §4.5 is that it provides a systematic means of (1) synthesizing a model, (2) ranking elements by relative importance, and (3) only including those elements necessary to sufficiently predict system dynamics.

In [16], Louca et al. have applied activity analysis to reduce an extensive vehicle model. The model includes detailed representation of the vehicle’s suspension and drive-train. Louca et al. have successfully derived a reduced vehicle model using activity analysis and they have experimental data to validate their results. However, they only show how an existing, validated model can be reduced top-down. What if a model did not exist? Given a predefined objective, how would one more systematically synthesize a vehicle model?

Take, for instance, the synthesis of a vehicle model. The objective is to synthesize a relatively simple model for predicting the vehicle speed and position under hard acceleration to top speed. The model will be used to examine the general effects of aerodynamic drag and rolling friction to help form the basis for the design of a new electric vehicle. Several elements are known to apply but their relative importance is not exactly known. These elements being the vehicle inertia $I_{\text{vehicle}}$, the wheels rotational inertia $I_{\text{wheels}}$, the aerodynamic drag $R_{\text{drag}}$, and the rolling friction $R_{\text{friction}}$. Still other
elements may apply. To insure a good approximation a 95% total activity threshold and 5% relative error tolerance will be employed. Since it is unclear what the relative importance of the necessary elements might be (i.e. drag or rolling friction might have a very low activity index, but may be necessary to predict vehicle dynamics) a rather liberal element activity threshold of 0.5% will be used to retain most elements of lower activity.

\[ F_{\text{aero}} \]

\[ W_v = m_v g \]

\[ R_w \]

\[ T_w \]

\[ F_f \]

\[ F \]

\[ x \]

\[ \phi \]

Figure 5.1: Vehicle schematic.

The vehicle mass \( m_v \) is known to be of greatest importance, plus it is of much greater mass than the wheels. So the initial model only includes vehicle inertia and the effort source inputs. The vehicle inertia is the only element necessary to predict the distance and speed. The sources are the motor torque \( T_m \) and the counteracting force due to the weight of the vehicle as it climbs an incline \( m_v g \sin \phi \) (where \( \phi \) is the angle of the incline). Two transformers must be included to account for the transmission gear ratio \( GR \) and the wheel radius \( R_w \) which transforms the torque into a force that
accelerates the vehicle. The initial model is shown in Figure 5.2 (a). The resulting system equation is

\[ m_v \ddot{x} = T_m \frac{GR}{R_w} - m_v g \sin \phi \]

This model includes a single energy storing element and so activity analysis cannot be applied. An element must be augmented.

\[ T_m \text{ GR} \quad \frac{1}{R_w} \]

\[ m_v \]

\[ E: m_v \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{drag}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

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\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

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\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

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\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

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\[ R: R_{\text{friction}} \]

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\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

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\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

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\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

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\[ E: m_v g \sin \phi \]

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\[ E: m_v g \sin \phi \]

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\[ m_v \]

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\[ m_v \]

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\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]

\[ R: R_{\text{friction}} \]

\[ m_v \]

\[ E: m_v g \sin \phi \]
model is given in bond graph form in Figure 5.2 (b) and the resulting equation is

\[ m_\nu \ddot{x} = T_m \frac{GR}{R_w} - m_\nu g \sin \phi - F_f \]

where\(^1\)

\[ F_f = \mu m_\nu g \cos \phi (1 + 3.1 \times 10^{-3} \dot{x} + 6.0 \times 10^{-5} \dot{x}^2) . \]

and where \( \mu \) is the coefficient of rolling friction. The element activities are calculated as

\[ A_{I\text{vehicle}} = \int_0^\tau |m_\nu \dot{x} \cdot \ddot{x}| \, dt \quad \text{and} \]

\[ A_{R\text{friction}} = \int_0^\tau |F_f \cdot \ddot{x}| \, dt . \]

The element activity indices and cumulative indices for each model are given in Table 5.1. As shown in the table the cumulative index for elements 1, \ldots, \( n-1 \) (\( CI^* \)) for the SECOND MODEL is less than the total activity threshold, and the activity index of the \( n \)th element (the rolling friction \( R_{friction} \)) is greater than the element activity threshold so it is retained. The algorithm then proceeds to augment another element.

The aerodynamic dynamic drag \( R_{drag} \) is augmented. The bond graph for this model is shown in Figure 5.2 (c) and its corresponding system equation is

\[ m_\nu \ddot{x} = T_m \frac{GR}{R_w} - m_\nu g \sin \phi - F_f - F_{aero} \]

where\(^2\)

\[ F_{aero} = \frac{1}{2} C_D A \rho \dot{x}^2 . \]

\(^1\)Note that the constitutive relation for the \( F_f \) is borrowed from Chang [4].

\(^2\)The constitutive relation for the aerodynamic drag can be found in [9].
<table>
<thead>
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<th>Rank</th>
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Table 5.1: Activity relative error results for iterative model synthesis.

$C_D$, $A$, and $\rho$ are the coefficient of drag, frontal area, and air density respectively. The aerodynamic drag element activity is

$$A_{R_{drag}} = \int_0^\tau |F_{aero} \cdot \dot{x}| \, dt.$$ 

As shown in Table 5.1 CI* for the **THIRD MODEL** is greater than threshold, but the mean relative errors on distance and speed between models two and three are greater
than tolerance. The \( n^{th} \) element \( (R_{\text{drag}}) \) has an activity index greater than the element activity threshold so it is retained.

The inertia due to the rotation of the wheels is then augmented to the model. The rotational inertia of each wheel is

\[
J = \frac{1}{2} m_w R_w^2
\]

and the combined inertia is \( 4J \). After simplification the vehicle model equation becomes

\[
(m_v + 2m_w)\ddot{x} = \frac{T_m}{R_w} - m_v g \sin \phi - F_{\text{aero}} - F_f. \tag{5.1}
\]

The wheel inertia activity is

\[
A_{I_{\text{wheels}}} = \int_0^\tau \left| 2m_w \dot{x} \cdot \ddot{x} \right| dt.
\]

The model meets both the total activity threshold and relative error tolerance criteria (refer to Table 5.1). Furthermore, this model corresponds well with models presented by Chang [4] and Karnopp [12] that have been extensively used for a variety of purposes. A bond graph representation of the final vehicle model is given in Figure 5.4 (d).

The model results are plotted in Figure 5.3. The output predictions increasingly converge with each successive model. The THIRD MODEL and FINAL MODEL match near exactly suggesting that THIRD MODEL is likely sufficient for the objectives given in problem statement.
5.2 Vehicle Mission Studies

The vehicle model can be thought of in a similar manner to that of the yo-yo model. Imagine that the model is used to simulate mission studies. For instance the model might be used to predict vehicle dynamics for a specific mission in which the vehicle accelerates to top speed, maintains that steady-state speed for a set distance, and then climbs a slight incline. Activity analysis could be used to examine the relative importance of elements during each stage of the mission – transient, steady-state, and incline. This analysis can be used to then synthesize reduced models for each stage. The reduced models only retain the necessary elements to sufficiently predict the model out-
put (e.g. enough elements to predict outputs to within 5% relative error). These models can be used to more efficiently represent the vehicle.

For this exercise the vehicle mission is: it begins on level ground (i.e. $\phi = 0$) from a stop and accelerates to top speed where it remains at steady-state. The vehicle parameters are as follows:³

\[
\begin{align*}
T_m &= 150 \text{ N-m} & \text{GR} &= 10.946 & m_v &= 1400 \text{ kg} \\
m_w &= 6.35 \text{ kg} & R_w &= 29.15 \text{ cm} & C_D &= 0.19 \\
A &= 2.26 \text{ m}^2 & \rho &= 1.225 \text{ kg/m}^3 & \mu &= 0.24
\end{align*}
\]

The Final Model (Equation 5.1) is employed for this exercise. Activity analysis is used to determine the relative importance of each model element during the initial transient and steady-state. For this exercise the Transient will be defined as $[0 \ 5\tau]$ and Steady-State as $[5\tau \ \tau_{final}]$. Note that Equation 5.1 is a first order differential equation in terms of $\dot{x}$ and that $\tau$ is the time constant corresponding to the moment at which the first-order response reaches roughly 63.2% of its steady-state value. $5\tau$ is chosen because it is the multiple of the time constant at which the transient reaches approximately 99% of the steady-state value.

Activity indices and cumulative indices for the Transient and Steady-State are tabulated in Table 5.2. For this exercise a 95% total activity threshold is used to simplify the existing model model into two sub-models to predict the transient and steady-

³Most parameters values used match those of the General Motors EV1 and can be found at the EV1 vehicle “Specs” and “Dimensions” web page [www.gmev.com/specs/specs.htm](http://www.gmev.com/specs/specs.htm).
<table>
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<tr>
<td>1</td>
<td>$R_{friction}$</td>
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</tr>
<tr>
<td>2</td>
<td>$R_{drag}$</td>
<td>0.1311</td>
</tr>
<tr>
<td>3</td>
<td>$I_{vehicle}$</td>
<td>0.0904</td>
</tr>
<tr>
<td>4</td>
<td>$I_{wheels}$</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

Table 5.2: Vehicle element activity indices and cumulative indices.

The Final Model includes 4 energy storing or dissipating elements: the vehicle inertia $I_{vehicle}$, the combined wheel inertia $I_{wheels}$, aerodynamic drag $R_{drag}$, and rolling friction $R_{friction}$.

As shown in the table, during the Transient the combined wheel inertia $I_{wheels}$ can be neglected. The resulting mathematical sub-model becomes

$$(m_v + 0)\ddot{x} = T_m \frac{GR}{R_w} - m_v g \sin \phi - F_{aero} - F_f$$

(5.2)

and its corresponding bond graph is shown in Figure 5.4 (b). To meet the threshold requirement at Steady-State it is not necessary to include the vehicle inertia $I_{vehicle}$.

For Steady-State the reduced sub-model (refer to Figure 5.4 (c)) is

$$0 = T_m \frac{GR}{R_w} - m_v g \sin \phi - F_{aero} - F_f .$$

(5.3)

The equation that results is an algebraic constraint and not a differential equation. The vehicle speed remains the same as it was at the end of the Transient. The reduced
model switches between Equation 5.2 to Equation 5.3 as the vehicle simulation transitions from Transient to Steady-State.

Figure 5.4: (a) Original model. (b) Reduced model for Transient \([0 \tau]\). (c) Reduced model for Steady-State \([\tau \tau_{final}]\).

Results are plotted in Figure 5.5 and show the original model prediction along with the variable fidelity model prediction. The variable fidelity results match the original model near exactly.

Let us look at a more extensive example. Now the vehicle accelerates to a steady-state speed, but after about 10 km the vehicle climbs an incline. Three major modes are identified in the vehicles dynamics: the Transient, Steady-State, and
INCLINE modes. Activity indices are calculated over 3 ranges. The first is for the TRANSIENT, $[0, 5\tau]$. The second is for steady state, $[5\tau, \tau^*]$, where $\tau^*$ is the moment in time that the vehicle reaches 10 km where it begins to climb the incline. The final range is for the incline, $[\tau^*, \tau_{\text{final}}]$. The results are given in Table 5.3. According to these results the models for the TRANSIENT and STEADY-STATE remain the same. The INCLINE mode like the TRANSIENT requires that the vehicle inertia be re-incorporated.

As evidenced by the plots in Figure 5.6 the reduced model matches the original with negligible error. The simulation makes the vehicle inertia $I_{\text{vehicle}}$ zero after the vehicle reaches 99% of its steady-state speed. As the vehicle reaches the incline at 10
Table 5.3: Vehicle element activity indices and cumulative indices for hard acceleration with incline.

km, the simulation re-enables the vehicle inertia. The variable fidelity results for this vehicle model match better than those for the clutched yo-yo. These results further substantiate that (1) a simulation can be broken down into variable modes (as is done with switched systems in VSMS) and (2) activity indices can be used as a metric to derive the reduced sub-models. Variable-fidelity modeling is a viable idea, and a method that combines VSMS techniques and activity analysis is an applicable approach.

5.3 Summary

This simple example serves to show the potential of applying these ideas to more complex higher fidelity models. A high fidelity vehicle model can be broken up into sub-models just like a variable structure system. Each of those sub-models is used to rep-
Figure 5.6: Variable fidelity results for hard acceleration followed by 5° incline.

represent the system during variable modes of the vehicle dynamics (e.g. steady-state, hard-acceleration, braking, hill climbing, etc.). Using activity analysis, each of those sub-models can be reduced and tailored to predict only the dynamics in each respective mode. In this way, the higher fidelity model is converted into simpler variable models (as is done in VSMS) that a simulation can “switch” between. This potentially can make the model more tractable by systematically simplifying the overall model. Plus, the model can be simulated more efficiently costing less time.

Imagine, for example, the model includes a detailed suspension sub-model. Part of the mission study is to predict dynamics as the vehicle drives in a straight line over a
relatively smooth flat surface. The activity indices for the spring compliance and shock absorber damping would be negligible. Instead of integrating the spring compliance state during this stage of the mission study, the spring output can be held and this value used to re-initiate integration of the spring state when the vehicle approaches a uneven surface.

There are expanded uses of the ideas and approaches presented in the previous chapters. These expanded purposes can be potentially more significant than modeling the switched-system, test cases. These ideas provide more systematic metrics for synthesizing models and measuring their fidelity. Additionally, they provide a means for tailoring a model for a specific purpose or to predict model outputs during a specific dynamic mode.
Chapter 6

Conclusions and Future Work

6.1 Summary

At the onset of this research several goals and objectives were defined. The goals of this research were threefold:

1. Develop a methodology to aid in the systematic synthesis of switched system models of desirable fidelity (Chapter 3),

2. Develop a metric for quantifying the fidelity of a switched system model (Chapter 4), and

3. Expand those ideas to aid in the development of a variable fidelity modeling approach to improve simulation efficiency (Chapter 5).
In Chapter 3, ideas were presented that focused on defining guidelines to aid a modeler in more systematically synthesizing models for switched systems. Several methods in Chapter 4 provide means for more objectively quantifying the relative fidelity of a switched-system model. The ideas presented in Chapters 3 and 4 are expanded in Chapter 5 and applied to synthesize a variable fidelity model of a simple ground vehicle.

Chapter 3 explores the formulation of a switched system as a variable structure system (VSS). By examining these systems in such a way, extensive tools and theory already developed in the areas of linear and nonlinear control can be more readily applied to model, simulate, and design switched systems. VSS theory was developed to design nonlinear controls of changing structure that improve system response and performance. A VSS is a system with a changing control structure, and a switched system is a system with a changing physical structure. Variable structure model synthesis (VSMS) examines switched systems as systems designed to take advantage of a changing physical structure. By doing so, switched systems can be paralleled with VSS controls and the theory developed for VSS can be applied to switched systems. It has been shown that VSMS can aid the user in dissecting the switched system to better understand and redesign it.

An algorithm is presented as a basic guideline for deconstructing a discontinuous switched system into continuous, variable-structure sub-models that parallel the variable structures of a VSS. In doing so, the algorithm opens the door to applying well developed tools for examining stability, predicting performance, and suggesting redesign.
It has been shown for the cases of the boost converter and hydraulic-ram pump that this approach allows one to determine key parameters and their general effects on system dynamics. VSMS was further demonstrated in the development of clutched yo-yo model that makes reasonable prediction of the yo-yo dynamics. The switching nature of this device suggested it as a candidate for VSMS.

The focus in Chapter 4 was developing metrics for quantifying the relative fidelity of a switched-system model. It was shown how, using experimental data or a validated high fidelity model, relative error can be used to quantify a models fidelity and more systematically determine if a model meets the problem statement objectives. The activity index approach developed by Louca et al. was expanded to nonlinear discontinuous systems. To date, most publications using activity as a metric focus on continuous and almost exclusively linear systems. Activity analysis has been applied to switched systems, and it has been shown how this analysis can be used to localize error in a VSMS model and simplify the switched-system model. An original algorithm was presented that can be used to aid a modeler in iteratively synthesizing switched-system models. The algorithm uses activity indices and relative error to more systematically decide if the model derived is appropriate and meets the predefined objectives. Further development is necessary to devise a more refined algorithm that can be automated.

This algorithm, as evidenced by Chapter 5, has potential for extended use in a more general variety of continuous-time problems. More specifically it is shown how, based on some specified objectives, the algorithm can be used to synthesize a vehicle
model. Inspired by Liaw [13], a variable structure formulation was used to derive a simulation that switches between reduced sub-models that emphasize the dynamics of each mode of the vehicle while it accelerates to top speed. Activity indices were used to derive the reduced model, and VSMS concepts were employed to formulate the overall model and to switch between sub-models.

Chapter 5 inspires thoughts of further development and expanded application. With the advent of fast, inexpensive workstations, virtual product development has become increasingly attractive. This is a modern trend in the aircraft and automotive industries. Simulations currently in development in these industries can involve hundreds of states and thousands of parameters. A variable model fidelity approach that employs activity analysis could go a long way to identifying key parameters during various phases of a vehicle’s dynamics and aiding the developer in design improvements. Furthermore, this approach could be automated and potentially advance simulation efficiency. The potential applications are endless.

6.2 Limitations

This study raises additional questions that motivate future research and development. Do these concepts have more extensive applications? Can these concepts and metrics be improved and further developed? Thus far, VSMS has only been applied to a small group of test problems where the switches were treated as ideal. Continued application
to more complex problems will likely lead to further innovations and refinement. The metrics presented in Chapter 4 seem intuitive and reasonable, but there should be further tests to confirm their validity and determine if they can be improved.

More specifically, the yo-yo model in Chapter 4 shows that in its current form the activity-based, model-reduction technique presented cannot be applied haphazardly. Some basic engineering sense must be employed. As the yo-yo example suggests, with a switched-system, it is necessary to identify the modes of not only the system but also its subsystems. The yo-yo has three modes: DESCENT, FREE-WHEEL, and ASCENT. However, the clutch subsystem has its own modes: ENGAGED and DISENGAGED. The yo-yo has two switch structures. One is due to the constraint imposed by the string. The second is due to the actuation of the brakes. The activity indices are calculated over various time spans that capture each mode. Care must be taken in determining these time spans because the indices are sensitive to time spans chosen. It was necessary for the case of the yo-yo to calculate indices for the time spans corresponding to the yo-yo’s modes and the time spans corresponding to the clutch’s modes.

Further work must be done in this area to formulate rules for properly identifying switched-system modes and their corresponding time spans. These rules can be applied to insure that the activity indices are properly calculated and that the appropriate indices are used to make decisions when reducing a switched-system model. Along with more advanced versions of the algorithms in Chapter 4, these rules can be applied to automate the derivation of reduced, switched-system models.
6.3 Dissipative Switches

The fact that the test cases were modeled by assuming that the switches could be treated as ideal brings up an important issue. How could VSMS be applied in cases in which the switches are dissipative and where this dissipation is significant enough to warrant inclusion in the model?

The VSMS models In Chapter 3 can be represented in a similar manner to the way sliding modes controls are. Take for example the boost converter. The VSMS representation is

\[ \dot{\lambda} = V_{in} - \frac{R}{L}\lambda - \alpha \left[ \frac{1}{C}q \right] \]

\[ q = \alpha \left[ \frac{1}{L}\lambda \right] - \frac{1}{R_{load}C}q \]

where

\[ \alpha = \begin{cases} 
0 & \text{when the transistor conducts} \\
1 & \text{when the diode conducts} 
\end{cases} \]

Imagine that the control signal that opens and closes the BJT is a square-wave current \( i(t) \) that fluctuates between +1 and −1 where \( i(t) = +1 \) when the BJT conducts and \( i(t) = -1 \) when it is closed. \( \alpha \) is a function of the current \( i(t) \) and can be represented as

\[ \alpha(i(t)) = \frac{1}{2}[1 + \text{sgn}(i(t))] \]

The switch representation is ideal. Using this representation, the BJT is either completely on or completely off and does not dissipate energy.

In sliding mode control, a control that uses the signum function (\( \text{sgn}(s) \) where \( s \) is the sliding surface) is commonly used. Though it results in “chatter” about the
switching surface, it requires less power than a control that employs say a hyperbolic tangent function (tanh) that is much smoother. If $\alpha$ were

$$\alpha(i(t)) = \frac{1}{2}[1 + \tanh(i(t))]$$

the switch would dissipate power. This suggests that it is possible to represent switched-systems with dissipative switches using VSMS. Further work in this area is necessary to develop a method for deriving the proper switch representation $\alpha$ that accounts for the switch dissipation.

### 6.4 Closing Comments

In summary, this dissertation presents a basic methodology for organizing the synthesis of switched system models. Metrics for measuring model fidelity that use relative error and activity indices are presented and employed to improve the modeling and design of the boost converter, hydraulic-ram pump, and clutched yo-yo. This research shows the potential use and extended application of these metrics and concepts. This is, however, only the foundation of a continued effort that will lead to a more refined and elegant methodology for synthesizing switched-system models.
Appendix A

The Hydraulic-Ram Pump: Model
Derivation and Experimentation

A.1 Pump Parameters

The model detailed in §3.3 has 3 energy storing elements and 2 dissipative elements each of which has associated parameters. In addition to the water density, \( \rho = 1000 \) kg/m\(^3\), the parameters necessary to calculate the fluid inertias are:

\[
L_{\text{pipe}} = 8.5 \text{ ft} = 2.59 \text{ m} \quad D_{\text{pipe}} = 1.5 \text{ in} = 0.0381 \text{ m}
\]

\[
L_{\text{hose}} = 20 \text{ ft} = 6.096 \text{ m} \quad D_{\text{hose}} = 0.5 \text{ in} = 0.0127 \text{ m}
\]
where $L_{\text{pipe}}$, $D_{\text{pipe}}$, $L_{\text{hose}}$, and $D_{\text{hose}}$ are the pipe length, pipe diameter, hose length, and hose diameter respectively. The inlet pipe and outlet hose fluid inertias are

$$I_{\text{inlet}} = \frac{\rho L_{\text{pipe}}}{A_{\text{pipe}}} = \frac{\rho L_{\text{pipe}}}{\pi D_{\text{pipe}}^2/4} = 2.272 \times 10^6 \text{ kg/m}^4$$

and

$$I_{\text{outlet}} = \frac{\rho L_{\text{hose}}}{A_{\text{hose}}} = \frac{\rho L_{\text{hose}}}{\pi D_{\text{hose}}^2/4} = 4.812 \times 10^7 \text{ kg/m}^4$$

respectively.

It is assumed for initial results that the chamber is completely filled with water and that no air is present. There is compliance due to fluid in the chamber and due to the flexible chamber walls. Given the pumping chamber length, $L_{pc} = 0.4572 \text{ m} \approx 18 \text{ in}$, and diameter, $D_{pc} = 3.81 \text{ cm} \approx 1.5 \text{ in}$, and the bulk modulus of elasticity of water, $\beta = 2.241 \times 10^9 \text{ Pa}$, the compliance due to the fluid is

$$C_1 = \frac{V_0}{\beta} = \frac{\pi D_{pc}^2 L_{pc}}{4\beta} = 2.326 \times 10^{-13} \text{ m}^3/\text{Pa}$$

where $V_0$ is the initial chamber volume. The chamber is made of PVC with a modulus of elasticity $E_{pc} = 2.758 \text{ GPa} \approx 400,000 \text{ psi}$ and Poisson ratio $\nu_{pc} = 0.37$.\(^1\) The chamber wall thickness $t_{pc}$ is 6.35 mm $\approx 0.25 \text{ in}$. The compliance due to the flexible chamber is

$$C_2 = \frac{V_0 D_{pc}}{E_{pc} t_{pc}} \left( 1 - \frac{\nu_{pc}}{2} \right) = 2.329 \times 10^{-12} \text{ m}^3/\text{Pa}$$.

\(^1\)The material properties for PVC can be found at the IPEX Incorporated process piping web site: www.ipexinc.com/proppng/ppng1.html#material.
The hose is connected in parallel to the pump and is also flexible. It is made of an elastomer that is much less rigid than PVC. Because the hose is in parallel with the pumping chamber its capacitance can be lumped with the chamber capacitance for simplicity. Its modulus of elasticity and Poisson's ratio used for the hose are $E_{\text{hose}} = 3 \text{ MPa} (\approx 440 \text{ psi})$ and $\nu_{\text{hose}} = 0.33$, and the wall thickness of the hose $t_{\text{hose}}$ is 3.175 mm ($\approx 0.125 \text{ in}$). The hose compliance is

$$C_3 = \nu_0 \frac{D_{\text{hose}}}{E_{\text{hose}} t_{\text{hose}}} \left(1 - \frac{\nu_{\text{hose}}}{2}\right) = 8.502 \times 10^{-10} \text{ m}^3 \text{ Pa}.$$  

The total compliance $C_{pc}$ is calculated as

$$C_{pc} = C_1 + C_2 + C_3 = 8.528 \times 10^{-10} \text{ m}^3 \text{ Pa}.$$ 

During the redesign, an air bladder was placed in the pumping chamber. The compliance due to the air bladder is significantly greater than $C_{pc}$. Also, the constitutive relation for the pumping chamber compliance becomes nonlinear. For a compliant element the constitutive relation is expressed in the form

$$e = \Phi_C(q)$$

where $e$ and $q$ are the generalized effort and displacement. Therefore, the chambers constitutive relation should be of the form

$$P_{pc} = \Phi_C(V_{pc}).$$

---

2The hose is assumed to be made of a commercial elastomer. Elastomers are typically not characterized like other engineering materials (i.e. metals and alloys), but nitrile rubber (NBR, acrylonitrile-butadiene rubber) is and its properties can be found at the MatWeb web site: www.matweb.com.
The chamber is partially filled with air and partially with water. Initially, the air bladder occupies the top portion of the volume of the chamber \( V_{\text{air}} = 4.34 \times 10^{-4} \text{ m}^3 \approx 31.8 \text{ in}^3 \) and has an initial length of 38.1 cm \((\approx 15 \text{ in})\). The remaining bottom part of the chamber is occupied by water with an initial volume of \( V_{\text{water}} = 8.69 \times 10^{-5} \text{ m}^3 (\approx 3.53 \text{ in}^3) \). The pressure at the bottom of the chamber is

\[
P_{\text{pc}} = P_{\text{air}} + \rho g h_{\text{water}}
\]

where \( P_{\text{air}} \) is the pressure of the air in the chamber and \( h_{\text{water}} \) is the height of the water in the chamber. According to Fox and McDonald [9], \( P_{\text{air}} \) can be determined using

\[
P_{\text{air}} = P_{\text{air}_0} \left( \frac{V_{\text{air}}}{\Delta V_{\text{air}}} \right) = P_{\text{air}_0} \left( \frac{V_{\text{air}}}{V_{\text{air}}_0 - V_{\text{pc}}} \right)
\]

where \( V_{\text{pc}} \) is the now he change in volume of water in the chamber and \( P_{\text{air}_0} \) is the initial pressure of air in the bladder (just above atmospheric). The water height \( h_{\text{water}} \) is also directly related to the change in volume of water in the chamber,

\[
\rho g h_{\text{water}} = \rho g \frac{\Delta V_{\text{water}}}{A_{\text{pc}}} = \frac{\rho g}{\pi D_{\text{pc}}^2/4} (V_{\text{water}_0} + V_{\text{pc}}).
\]

Putting it all together, the constitutive relation used in the model for the pumping chamber when the air bladder is present is

\[
P_{\text{pc}} = P_{\text{air}_0} \left( \frac{V_{\text{air}}}{V_{\text{air}}_0 - V_{\text{pc}}} \right) + \frac{\rho g}{\pi D_{\text{pc}}^2/4} (V_{\text{water}_0} + V_{\text{pc}}).
\]

The elements left to parameterize are the pipe and hose resistances. The two are detailed in the following section.
A.2 Pipe and Hose Losses

Losses in the pump occur due to friction effects along the walls of the inlet pipe and outlet hose plus restrictions due to fittings, entrances, changing areas, etc. The major losses are due the friction affects along the walls of the inlet pipe and outlet hose. These losses can be accounted for by a friction factor as shown in §3.3 and as is repeated hear:

\[ \Delta P_{loss} = \frac{1}{2} f(Re) \rho \frac{L}{D} \frac{Q}{A} \frac{|Q|}{A} . \]

The friction factor \( f \) is generally a function of Reynolds number \( Re = \rho \bar{V}D/\mu = \rho (Q/A)D/\mu \). Since the fluid flow rate through the pump dynamically changes, it is necessary to have a mathematical correlation that accounts for the change in friction.

For laminar flow in a pipe under standard conditions (i.e. \( Re < 2300 \)) the friction factor is

\[ f_{laminar} = \frac{64}{Re} \]

The resulting pressure loss in the pipe is

\[ \Delta P_{loss} = \frac{1}{2} \left( \frac{64}{Re} \right) \rho \frac{L}{D} \frac{Q}{A} \frac{|Q|}{A} = \frac{1}{2} \left( \frac{64 \mu}{\rho Q D} \right) \rho \frac{L}{D} \frac{Q}{A} \frac{|Q|}{A} = \frac{32 \mu L}{D^2} \frac{|Q|}{A} \]

and can be further simplified as follows if the water only flows in a positive direction:

\[ \Delta P_{loss} = \frac{32 \mu L}{D^2} \left( \frac{Q}{A} \right) = \frac{128 \mu L}{\pi D^4} Q . \]

For laminar flow, the pipe resistance can then be expressed as

\[ R_{inlet} = \frac{\Delta P_{inlet}}{Q_{in}} = \frac{128 \mu L_{pipe}}{\pi D_{pipe}^4} \] (A.1)
and the hose resistance as

\[ R_{\text{outlet}} = \frac{\Delta P_{\text{outlet}}}{Q_{\text{out}}} = \frac{128 \mu L_{\text{hose}}}{\pi D_{\text{hose}}^4}. \quad (A.2) \]

The resulting constitutive relations are linear.

For turbulent flow (i.e. \( Re > 2300 \)) the friction factors are determined experimentally and can be found plotted in a Moody diagram. However, as is mentioned in Fox and McDonald [9], a mathematical formulation for the friction factor in terms of Reynolds number is necessary to facilitate computer simulations. For this purpose, the Blassius correlation for turbulent flow in a smooth pipe can be used and is generally valid for \( Re \leq 10^5 \). The Blassius correlation is

\[ f_{\text{turbulent}} = \frac{0.3164}{Re^{0.25}} \]

and the resulting pressure loss

\[ \Delta P_{\text{loss}} = \frac{1}{2} \left( \frac{0.3164}{Re^{0.25}} \right) \frac{L}{D} \frac{Q}{A} \frac{Q}{A}. \]

Unlike the laminar flow correlation, the turbulent correlation is generally nonlinear. For turbulent flow, the pipe and hose resistance are

\[ R_{\text{inlet}} = \frac{1}{2} \left( \frac{0.3164}{Re_{\text{inlet}}^{0.25}} \right) \frac{L_{\text{pipe}}}{D_{\text{pipe}} A_{\text{pipe}}^2} \left| Q_{\text{in}} \right| = \left( \frac{0.3164}{Re_{\text{inlet}}^{0.25}} \right) \frac{2L_{\text{pipe}}}{\pi D_{\text{pipe}}^3} \left| \Gamma_{\text{in}} \right| \quad (A.3) \]

and

\[ R_{\text{outlet}} = \left( \frac{0.3164}{Re_{\text{outlet}}^{0.25}} \right) \frac{L_{\text{hose}}}{D_{\text{hose}} A_{\text{hose}}^2} \left| Q_{\text{out}} \right| = \left( \frac{0.3164}{Re_{\text{outlet}}^{0.25}} \right) \frac{2L_{\text{hose}}}{\pi D_{\text{hose}}^3} \left| \Gamma_{\text{out}} \right| \quad (A.4) \]
respectively.

Minor losses are due to restrictions and can be expressed as

\[ \Delta P_{in} = K \frac{Q}{A} \left| \frac{Q}{A} \right| \]

where \( K \) is the loss coefficient and must be determined experimentally. Since this would be cumbersome and likely beyond the scope necessary for the pump model, these losses are assumed to be relatively negligible.

Equations A.1-A.4 are used to model the pressure loss in the pipe and hose. The simulation calculates the Reynolds number at each time step and switches to the appropriate constitutive law based on the Reynolds number at that time step.

### A.3 Experimental Setup

To validate the pump model a hydraulic-ram pump was purchased and setup in the laboratory. The model is capable of predicting pressures at junctions 1 and 2 and the inlet and outlet flow rates. The sensors used output a voltage signal that is ported into a data acquisition board connected to a workstation as is shown in Figure A.1. The experimental results are logged and saved using a custom built LABVIEW virtual instrument.

The junction pressures are measured by pressure transducers that convert pressure to an output voltage signal. Each transducer was individually calibrated against a pressure gauge using a variable pressure source. The output voltages were correlated
with known pressures.

The inlet and outlet flow rates presented challenges. Most, affordable commercial flow meters are designed to capture flow rate at steady-state. After some research and much consideration, a paddle wheel flow sensor was chosen as an affordable option. It was used at the inlet to measure the inlet flow rate. The sensor used is limited to flow measures of 1 to 20 ft/s (which for a 1.5 in diameter pipe corresponds to approximately 5.4 to 108.4 GPM). It was found that the sensor had a limited response time and that it was not sufficiently responsive to capture the dynamics of the inlet flow rate.
Optional sensors were more expensive and had only marginally improved response.

Figure A.2: Waste-gate valve motion.

The waste-gate valve vertical motion is tracked using a *linear variable differential transformer* (LVDT). This data is used to substantiate the on/off logic used in the model to approximate the valve motion. Examination of the valve motion (refer to Figure A.2) shows that the waste-gate valve is near fully-closed for approximately 25% of the cycle. To simplify analysis and facilitate the VSS model formulation used in Chapter 3, it is assumed that the valve actuates like an ideal switch so that its motion can be approximated by a square wave with a 25% duty-cycle as shown in Figure A.2.
Appendix B

Yo-Yo Model Derivation

B.1 Simple Yo-Yo Model

A Lagrange approach was used to derive the yo-yo model. The first step is to determine the kinetic energy. The simple yo-yo stores kinetic energy by means of rotation and vertical translation. The kinetic energy is

\[ T_{\dot{y} \dot{\theta}} = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} J \dot{\theta}^2 \]

where \( M \) is the total yo-yo mass and \( J \) is its rotational inertia. The potential energy associated with the yo-yo is

\[ V_y = Mgy. \]

The Lagrange function \( L \) is

\[ L = T - V. \]
The yo-yo rotates with no slip relative to the string imposing an holonomic constraint. The constraint, though, changes. When the yo-yo is initially descending and the string is right of center (Structure A), as in Figure 3.17 (a), the torque due to the string tension is positive and the no-slip constraint and its derivatives are

\[ R_{\text{string}} \theta = H - y, \]
\[ R_{\text{string}} \dot{\theta} = -\dot{y}, \text{ and} \]
\[ R_{\text{string}} \ddot{\theta} = -\ddot{y} \]  

(B.1)

where \( R_{\text{string}} \) and \( H \) are the radius at which the spring tension is applied and the height from which the yo-yo is dropped. Due to the constraint the system has only one independent generalized coordinate, \( \theta \). As the yo-yo ascends and string is left of center (Structure B), as in Figure 3.17 (b), the applied torque is negative and constraint and its derivatives become

\[ R_{\text{string}} \left( \theta - \frac{L_{\text{string}}}{R_{\text{string}}} \right) = y - (H - L_{\text{string}}), \]
\[ R_{\text{string}} \dot{\theta} = \dot{y}, \text{ and} \]
\[ R_{\text{string}} \ddot{\theta} = \ddot{y}. \]  

(B.2)

\( L_{\text{string}} \) is the unraveled length of the yo-yo string.

The system equations are determined for each stage of the yo-yo motion. Using the constraint in Equation B.1 the kinetic and potential energies for Structure A are

\[ T_\theta = \frac{1}{2} M(-R_{\text{string}} \dot{\theta})^2 + \frac{1}{2} J\dot{\theta}^2 = \frac{1}{2} MR_{\text{string}}^2 \dot{\theta}^2 + \frac{1}{2} J\dot{\theta}^2 \]

and

\[ V_y = Mg(H - R_{\text{string}} \theta) = MgH - MgR_{\text{string}} \theta. \]
One last thing is necessary to derive the system equations – the nonconservative forces. The nonconservative forces $Q_i$ are due the string tension $T_{string}$ and the string coulomb friction. The string rubs against the inner walls of the yo-yo causing a dissipative friction force of the form $\mu N sgn(\dot{\theta})$. The virtual work $\delta W$ due to the generalized forces is

$$\delta W = T_{string} \delta y + T_{string} R_{string} \delta \theta - \mu N sgn(\dot{\theta}) \cdot \delta \theta$$

$$= T_{string} \delta (H - R_{string}\theta) + T_{string} R_{string} \delta \theta - \mu N sgn(\dot{\theta}) \cdot \delta \theta$$

$$= -T_{string} R_{string} \delta \theta + T_{string} R_{string} \delta \theta - \mu N sgn(\dot{\theta}) \cdot \delta \theta.$$  

The second term in the virtual work is positive because the string is positioned relative to the yo-yo center such that it imparts a positive torque. Note that the virtual work must be derived in terms of the independent generalized coordinate(s). Using Lagrange’s equation,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = Q_\theta,$$

the system equation is

$$MR_{string}^2 \ddot{\theta} + J\ddot{\theta} - MgR_{string} = -T_{string} R_{string} + T_{string} R_{string} - \mu N sgn(\dot{\theta})$$

$$= -\mu N sgn(\dot{\theta}).$$  

(B.3)

An alternate form of the above equation can be derived using the summation of forces on the yo-yo in the vertical direction:

$$\sum F_y = M\ddot{y} = T_{\text{string}} - Mg.$$  

(B.4)

Recall that for Structure A the no-slip constraint dictates that $R_{string}\dot{\theta} = -\dot{y}$. Substitut-
ing this relation into the above equation and multiplying through by $R_{\text{string}}$ yields

$$-M R_{\text{string}}^2 \ddot{\theta} = T_{\text{string}} R_{\text{string}} - M g R_{\text{string}}$$

which can be substituted into Equation B.3. After some simplification the result is

$$J \ddot{\theta} = R_{\text{string}} T_{\text{string}} - \mu N \text{sgn}(\dot{\theta}) . \quad (B.5)$$

Equation B.5 along with Equation B.4 can be used as an alternative to Equation B.3. Note that this result is the same as that that can be achieved without assuming the no-slip constraint.

The advantage of Equation B.3 is that it is independent of the string tension $T_{\text{string}}$ which is useful because the string tension is not a source and is not directly known. However, Equations B.4 and B.5 allow one to more clearly identify the term associated with the yo-yo vertical translation versus that associated with its rotation. For instance, $J \ddot{\theta}$ is associated with yo-yo rotation while

$$M R_{\text{string}}^2 \dot{\theta} = -M \ddot{y} \cdot R_{\text{string}}$$

is associated with the vertical translation. This differentiation is more important for the clutched yo-yo model to facilitate the activity analysis applied in Chapter 4.

In much the same manner the equations for Structure B can be derived. Though the constraint equation changes, the kinetic energy remains the same:

$$T_\theta = \frac{1}{2} M (R_{\text{string}} \dot{\theta})^2 + \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} M R_{\text{string}}^2 \dot{\theta}^2 + \frac{1}{2} J \dot{\theta}^2 .$$
The potential energy does change, though:

\[ V_y = Mg \left[ R_{\text{string}} \left( \theta - \frac{L_{\text{string}}}{R_{\text{string}}} \right) + (H - L_{\text{string}}) \right]. \]

Since the constraint equation changes and the string tension now imparts a negative torque the virtual work is

\[
\delta W = T_{\text{string}} \delta y - T_{\text{string}} R_{\text{string}} \delta \theta - \mu N \text{sgn}(\dot{\theta}) \cdot \delta \theta \\
= T_{\text{string}} \delta \left[ R_{\text{string}} \left( \theta - \frac{L_{\text{string}}}{R_{\text{string}}} \right) + (H - L_{\text{string}}) \right] - T_{\text{string}} R_{\text{string}} \delta \theta - \mu N \text{sgn}(\dot{\theta}) \cdot \delta \theta \\
= T_{\text{string}} R_{\text{string}} \delta \theta - T_{\text{string}} R_{\text{string}} \delta \theta - \mu N \text{sgn}(\dot{\theta}) \cdot \delta \theta.
\]

The second term in the virtual work is negative because the string is left of center and imparts a negative torque. After applying Lagrange’s equation the resulting system equation is

\[
MR_{\text{string}}^2 \ddot{\theta} + J\ddot{\theta} + MgR_{\text{string}} = T_{\text{string}} R_{\text{string}} - T_{\text{string}} R_{\text{string}} - \mu N \text{sgn}(\dot{\theta}) \quad \text{(B.6)}
\]

As above, Equation B.4, along with the Structure B constraint equation (Equation B.2) can be used to derive an alternative version of Equation B.6:

\[
MR_{\text{string}}^2 \ddot{\theta} = T_{\text{string}} R_{\text{string}} - MgR_{\text{string}}.
\]

By substituting this into Equation B.6 the alternative is derived:

\[
J\ddot{\theta} = -T_{\text{string}} R_{\text{string}} - \mu N \text{sgn}(\dot{\theta}) \quad \text{(B.7)}
\]
Using a variable structure formulation the yo-yo model can be represented as a single set of equations. The system equations used in Chapter 3 are

\[ J\ddot{\theta} = \alpha T_{\text{string}} R_{\text{string}} - \mu N \text{sgn}(\dot{\theta}) \]  \hspace{1cm} (B.8)

and

\[ M\ddot{y} = T_{\text{string}} - Mg \]  \hspace{1cm} (B.9)

where

\[ \alpha = \begin{cases} 
-1 \text{ when } \theta < L_{\text{string}} \frac{R_s}{L_{\text{string}}} & , \\
+1 \text{ when } \theta > L_{\text{string}} \frac{R_s}{L_{\text{string}}} 
\end{cases} \]

This formulation is used as opposed to Equations B.3 and B.6 because it is a bit more intuitive and because these equations allow one to better distinguish terms for activity analysis. Additionally, this formulation matches that derived by a more common kinematics approach.

\section*{B.2 Clutched Yo-Yo Model}

The derivation of the clutched yo-yo model is much more involved than the simple yo-yo model. The brake masses introduce additional energy storing elements so the kinetic energy is

\[ T_{y\dot{\theta}r_1r_2r_3} = \frac{1}{2}(M-3m)\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{r}_1 \cdot \dot{r}_1 + \frac{1}{2}m\dot{r}_2 \cdot \dot{r}_2 + \frac{1}{2}m\dot{r}_3 \cdot \dot{r}_3 \]

\[ = \frac{1}{2}(M-3m)\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{3}{2}m(\dot{r}_1 \cdot \dot{r}_1 + \dot{r}_2 \cdot \dot{r}_2 + \dot{r}_3 \cdot \dot{r}_3) \]
where \( \mathbf{r}_1, \mathbf{r}_2, \) and \( \mathbf{r}_3 \) are the vector velocities of each brake mass \( m \) relative to a fixed point on the ground directly below the yo-yo. The brakes are set \( \frac{2\pi}{3} \) rad apart so their relative vector positions and velocities are

\[
\mathbf{r}_1 = y\mathbf{i} + r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j},
\]
\[
\mathbf{r}_2 = y\mathbf{i} + r\cos\left(\theta + \frac{2\pi}{3}\right)i + r\sin\left(\theta + \frac{2\pi}{3}\right)\mathbf{j},
\]
\[
\mathbf{r}_3 = y\mathbf{i} + r\cos\left(\theta + \frac{4\pi}{3}\right)i + r\sin\left(\theta + \frac{4\pi}{3}\right)\mathbf{j},
\]

and

\[
\dot{\mathbf{r}}_1 = (\dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta})\mathbf{i} + (\dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta} + y)\mathbf{j},
\]
\[
\dot{\mathbf{r}}_2 = \left[\dot{r}\cos\left(\theta + \frac{2\pi}{3}\right) - r\sin\left(\theta + \frac{2\pi}{3}\right) \cdot \dot{\theta}\right]\mathbf{i} +
\left[\dot{r}\sin\left(\theta + \frac{2\pi}{3}\right) + r\cos\left(\theta + \frac{2\pi}{3}\right) \cdot \dot{\theta} + \dot{y}\right]\mathbf{j},
\]
\[
\dot{\mathbf{r}}_3 = \left[\dot{r}\cos\left(\theta + \frac{4\pi}{3}\right) - r\sin\left(\theta + \frac{4\pi}{3}\right) \cdot \dot{\theta}\right]\mathbf{i} +
\left[\dot{r}\sin\left(\theta + \frac{4\pi}{3}\right) + r\cos\left(\theta + \frac{4\pi}{3}\right) \cdot \dot{\theta} + \dot{y}\right]\mathbf{j},
\]

where \( r \) and \( \dot{r} \) are the radial position and radial velocity of the brakes. The brakes are assumed to move out radially at the same rate. Using the above vector velocities, and some trigonometric identities the third term in the kinetic energy can be derived. After some manipulation and simplification the third term is

\[
\frac{3}{2}m (\mathbf{r}_1 \cdot \dot{\mathbf{r}}_1 + \mathbf{r}_2 \cdot \dot{\mathbf{r}}_2 + \mathbf{r}_3 \cdot \dot{\mathbf{r}}_3) = \frac{3}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{y}^2)
\]

making the kinetic energy

\[
T_{\dot{y}\dot{\theta}rr} = \frac{1}{2}(M - 3m)\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{3}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{y}^2)
\]
\[
= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{3}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2).
\]
The springs in the clutched yo-yo introduce additional energy storing elements accounted for in the potential energy. The potential energy is

\[ V_{yr} = Mg y + 3 \left[ \frac{1}{2} k (r - r_0)^2 \right] . \]

The constraint equations for Structure A and B remain as defined earlier. However, an additional structure is introduced, Structure C – the yo-yo free-wheeling. As the yo-yo free-wheels it does not translate vertically. The constraint is then

\[ y = \dot{y} = \ddot{y} = 0. \]  \hspace{1cm} (B.10)

This third structure also introduces an added nonconservative force; the additional force \(-B \dot{\theta}\) is due to the damping that occurs when the yo-yo free-wheels.

Structure A is first examined. Applying the constraint in Equation B.1 the kinetic energy can be re-written as

\[
T_{\theta r \dot{r}} = \frac{1}{2} M (\dot{R}_{string} \dot{\theta})^2 + \frac{1}{2} J \dot{\theta}^2 + \frac{3}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)
\]

\[ = \frac{1}{2} MR_{string}^2 \dot{\theta}^2 + \frac{1}{2} J \dot{\theta}^2 + \frac{3}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \]

and the potential energy as

\[ V_{yr} = Mg (H - R_{string} \dot{\theta}) + 3 \left[ \frac{1}{2} k (r - r_0)^2 \right] \]

\[ = Mg H - Mg R_{string} \dot{\theta} + \frac{3}{2} k (r - r_0)^2 . \]

Applying Lagrange’s equation gives the following set of system equations:

\[ MR_{string}^2 \ddot{\theta} + J \ddot{\theta} + (3mr^2 \ddot{\theta} + 6mr \dot{\theta}) - Mg R_{string} = -\mu N \text{sgn} (\dot{\theta}) \text{ and } \]

\[ m \dddot{r} = m (\dot{r}^2 - k (r - r_0)). \]  \hspace{1cm} (B.11)
As was done in the previous section Equation B.4 can be used to derive an alternative form of Equation B.11. The alternative set of equations is

\[
J\ddot{\theta} + (3mr^2\ddot{\theta} + 6mr\dot{r}\dot{\theta}) = T_{\text{string}}R_{\text{string}} - \mu N \text{sgn} (\dot{\theta}) ,
\]

\[
M\ddot{y} = T_{\text{string}} - Mg , \text{ and }
\]

\[
m\ddot{r} = m\dot{r}\dot{\theta}^2 - k(r - r_0) .
\]

Next, Structure C (free-wheeling) is evaluated. By applying the constraint in Equation B.10 the kinetic and potential energies are

\[
T \dot{\theta} r = \frac{1}{2} J\dot{\theta}^2 + \frac{3}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)
\]

\[
= \frac{1}{2} J\dot{\theta}^2 + \frac{3}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)
\]

and

\[
V_{yr} = \frac{3}{2} k(r - r_0)^2 .
\]

After applying Lagrange’s equation the set of system equations becomes

\[
J\ddot{\theta} + (3mr^2\ddot{\theta} + 6mr\dot{r}\dot{\theta}) = -B\dot{\theta} \quad \text{and} \quad \dot{B} = 0, \quad \dot{M} = 0
\]

(\text{B.13})

\[
m\ddot{r} = m\dot{r}\dot{\theta}^2 - k(r - r_0) .
\]

Last, Structure B is examined. The constraints in Equations B.2 are utilized to determine the kinetic and potential energies,

\[
T \dot{\theta} r = \frac{1}{2} M(R_{\text{string}}\dot{\theta})^2 + \frac{1}{2} J\dot{\theta}^2 + \frac{3}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)
\]

\[
= \frac{1}{2} MR_{\text{string}}^2 \dot{\theta}^2 + \frac{1}{2} J\dot{\theta}^2 + \frac{3}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)
\]

and

\[
V_{yr} = Mg \left[ R_{\text{string}} \left( \theta - \frac{L_{\text{string}}}{R_{\text{string}}} \right) + (H - L_{\text{string}}) \right] + 3 \left[ \frac{1}{2} k(r - r_0)^2 \right] .
\]
The system equations are then
\[ MR_{\text{string}}^2 \ddot{\theta} + J \ddot{\theta} + (3mr^2 \dot{\theta} + 6mr \dot{r} \dot{\theta}) + MgR_{\text{string}} = -\mu N \text{sgn} (\dot{\theta}) \quad \text{and} \quad (B.14) \]
\[ m \ddot{r} = m \dot{\theta}^2 - k(r - r_0). \]

Alternatively, the following equations can be used:
\[ J \ddot{\theta} + (3mr^2 \ddot{\theta} + 6mr \dot{r} \dot{\theta}) = -T_{\text{string}} R_{\text{string}} - \mu N \text{sgn} (\dot{\theta}), \]
\[ M \ddot{y} = T_{\text{string}} - Mg, \quad \text{and} \quad (B.15) \]
\[ m \ddot{r} = m \dot{\theta}^2 - k(r - r_0). \]

A variable structure formulation is employed along with Equations B.12, B.13, and B.15 to produce one concise set of equations to describe the yo-yo’s dynamics. The VSMS model used in Chapter 3 is
\[ (J + 3mr^2) \ddot{\theta} + 6mr \dot{r} \dot{\theta} = \alpha_1 R_{\text{string}} T_{\text{string}} - (1 - \alpha_2) \mu N \text{sgn} (\dot{\theta}) - \alpha_2 B \dot{\theta} \quad (B.16) \]
\[ M \ddot{y} = T_{\text{string}} - Mg \quad (B.17) \]
and
\[ m \ddot{\theta} = m \dot{\theta}^2 - k(r - r_0) \quad (B.18) \]

where
\[ \alpha_1 = \begin{cases} 
-1 & \text{when string is left of center of mass} \\
0 & \text{when the yo-yo free-wheels at the end of the string} \\
+1 & \text{when string is right of center of mass} 
\end{cases} \]

and where
\[ \alpha_2 = \begin{cases} 
0 & \text{when the yo-yo is descending or ascending, } y > H - L_{\text{string}} \\
1 & \text{when the yo-yo free-wheels at the end of the string, } y = H - L_{\text{string}} 
\end{cases}. \]
As mentioned in the previous section, the reason for using the alternative forms is to aid in distinguishing terms for activity analysis. The generalized momentum \( p_i \) associated with each independent generalized coordinate is defined as

\[
p_i = \frac{\partial T}{\partial \dot{q}_i}.
\]

The momenta are thus

\[
p_\theta = J\dot{\theta} + 3mr^2\ddot{\theta}, \quad \text{(B.19)}
\]

\[
= p_{\theta yo-yo} + p_{\theta brakes},
\]

\[
p_y = M\ddot{y}, \quad \text{and} \quad \text{(B.20)}
\]

\[
p_r = 3m\ddot{r}. \quad \text{(B.21)}
\]

Equation B.19 is made of two terms: (1) the angular momentum due to the yo-yo \( p_{\theta yo-yo} \) and (2) the angular momentum due to the brake masses \( p_{\theta brakes} \).

The purpose for identifying these momenta is to facilitate the derivation of element activities for Chapter 4. The generalized momenta can be divided into 4 components associated with the yo-yo rotational inertia, yo-yo vertical translational inertia, brake radial translational inertia, and brake rotational inertia. The activities associated with these elements are

\[
A_{l yo-yo rotation} = \int_0^\tau \left| \dot{p}_{\theta yo-yo} \cdot \frac{p_{\theta yo-yo}}{J} \right| dt = \int_0^\tau \left| J\ddot{\theta} \cdot \dot{\theta} \right| dt,
\]

\[
A_{l yo-yo vertical translation} = \int_0^\tau \left| \dot{p}_y \cdot \frac{p_y}{M} \right| dt = \int_0^\tau \left| M\ddot{y} \cdot \dot{y} \right| dt,
\]

\[
A_{l brake radial translation} = \int_0^\tau \left| \dot{r} \cdot \frac{p_r}{3m} \right| dt = \int_0^\tau \left| 3m_{brake} r^2 \ddot{\theta} \right| dt, \quad \text{and}
\]

\[
A_{l brake rotation} = \int_0^\tau \left| \dot{p}_{\theta brakes} \cdot \frac{p_{\theta brakes}}{3mr^2} \right| dt = \int_0^\tau \left| (3m_{brake} r^2 \ddot{\theta} + 6m_{brake} r\ddot{r} \cdot \dot{\theta}) \cdot \dot{\theta} \right| dt.
\]
The final energy storing element is associated with the springs. The effort associated with the springs is \(3k(r - r_0)\). Thus the spring activity is

\[
A_{C_{\text{spring}}} = \int_0^\tau |3k(r - r_0) \cdot \dot{r}| \, dt,
\]

leaving only the dissipative elements associated with the stick-slip friction \((\mu N \text{sgn}(\dot{\theta}))\) and viscous damping \((B\dot{\theta})\). Their relative activities are

\[
A_{R_{\text{stick-slip friction}}} = \int_0^\tau |\mu N \text{sgn}(\dot{\theta}) \cdot \dot{\theta}| \, dt, \quad \text{and}
\]

\[
A_{R_{\text{damping}}} = \int_0^\tau |B\dot{\theta} \cdot \dot{\theta}| \, dt.
\]

These activities are used in §4.4.2 and §4.4.3.
Bibliography


Vita

Javier Angel Kypuros was born in San Antonio, Texas on July 12, 1974, the son of Jose Jesus Kypuros and Mari Elena Kypuros. After graduating as valedictorian from Holy Cross High School, San Antonio, Texas, in 1992, he entered Princeton University in Princeton, New Jersey. He received the degree of Bachelor of Science in Engineering from Princeton University on June 4, 1996.


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