Combinatorial Analysis

Counting Principles:

When an experiment $E$ consists of an experiment $F$ with $m$ outcomes followed by an experiment $G$ with $n$ outcomes, the number of outcomes in $E$ is the product term $(mn)$.

Given $r$ experiments, each with $n_i$ possible outcomes. The total no. of outcomes is:

$$\prod_{i=1}^{r} n_i$$

Ex: Three ordinary dice are rolled. How many outcomes are possible? $(6^3)$
Permutations:

Arrangements where order is important, i.e., objects are unique.

No. distinct arrangements of items in a sequence of $n = n!$.

Suppose that select only $k$ of $n$ available objects to form a sequence. That is, the no. of permutations of $n$ objects $k$ at a time. The total no. of such sequences is:

$$nP_k = \binom{n}{k} = \frac{n!}{(n-k)!} = n(n-1)(n-2) \cdots (n-k+1) = n!/(n-k)!$$

Ex: Select 3 students from a committee of 10 to serve as officers. $(10)_3 = 10 \cdot 9 \cdot 8$

Now suppose that some of the objects are indistinguishable (e.g., $k$ blocks of one color and $(n-k)$ blocks of another color). $n!$ is too large.
**Combinations:**

Permutations where order is not important or items are not unique. Call this \( \binom{n}{k} \)

Consider this as a composite experiment. First form a color pattern, then reveal the distinct identities of the objects, to form one of the one of the \( n! \) arrangements. The count \( n! \) is the no of color patterns multiplied by the number of ways of arranging the \( k \) objects of one color and the \( n-k \) objects of the other color.

\[
n! = \binom{n}{k} \cdot k! \cdot (n-k)!
\]

So

\[
\binom{n}{k} = \left( \frac{n!}{k!(n-k)!} \right)
\]

Ex: Choosing a subcommittee of 3 from a group of 10:

\[
\binom{10}{3} = \left( \frac{10!}{3!(7)!} \right)
\]
Concept extends to:

\[ \frac{n!}{k_1!k_2!k_3!\ldots k_r!} \]

where \(k_1, k_2, k_3, \text{ etc of } n\) are indistinguishable.

Ex: How many ways can 12 persons at a bridge party be divided into 3 tables of 4?

a) Suppose that the tables can be distinguished:

\[ 12!/(4!4!4!) \]

or alternatively consider this as a sequence of 2 selections:

\[ \binom{12}{4} \binom{8}{4} \]

b) Now suppose that the tables are interchangeable and we are only interested in which 4 people are playing together at each table. The number must be divided by 3!, the no of ways of arranging the tables in sequence.
Ex: Consider n antennas - m defective, n-m functional, where individuals are not distinguishable within groups. How many linear orderings are possible where no two defectives are consecutive? i.e. n-m functional with at most 1 defective between a pair, ie n-m+1 positions for the m defectives. Select m of these positions.

\[
\binom{n-m+1}{m}
\]
Example

4 EE Books
3 Math Books
3 Reliability Books

Compute the number of Arrangements if

a) Books of a subject must stay together and each book is unique - $3!4!3!3!$

b) Only math books must stay together and each book is unique - 8 groups (7 of 1 and 1 of 3) $8!3!$

Stage 1: Arrange subject groups
2: Arrange books within subject group
Now, note for $1 \leq r \leq n$ the binomial coefficient is:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$
Binomial Theorem

\[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

Multinomial Theorem

\[(x_1 + x_2 + \ldots + x_r)^n = \sum_{(n_1, n_2, \ldots, n_r)=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}\]

Sum over all nonnegative integer valued vectors

\[\left( n_1 n_2 \ldots n_r \right)\]

such that

\[\sum_{i=1}^{r} n_i = n\]
Distribution of Balls in Urns

Given $r$ urns and $n \geq r$ balls. Distribute the balls

$$(r \cdot r \cdot r \cdots r) = r^n$$

Now suppose that the balls are not distinguishable - ie $n$ objects just divided into $r$ nonempty groups. This can be considered as a variation on the antenna problem - ie select $r-1$ of $n-1$ spaces between adjacent objects as dividing points. Let $x_i$ be the no of balls in urn $i$ and

$$\sum x_i = n$$

$$\left( \frac{(n-1)!}{(r-1)!(n-r)!} \right)$$

However, suppose that all urns do not necessarily contain a ball. Define $y_i = x_i + 1$. The no of distinct nonnegative integer-valued vectors ($x_1, x_2, ..., x_r$) is

$$\left( \frac{n+r-1}{r-1} \right)$$