OPTIMIZATION OF TRANSIENT HEATER SETTINGS TO PROVIDE SPATIALLY UNIFORM HEATING IN MANUFACTURING PROCESSES INVOLVING RADIANT HEATING

K. J. Daun, J. R. Howell, and D. P. Morton
Department of Mechanical Engineering, The University of Texas at Austin, Austin, Texas, USA

This article presents an optimization methodology for finding the heater settings that provide spatially uniform transient heating in manufacturing processes involving radiant heating. Equations governing the transient temperature and temperature sensitivity distributions over the product are first derived using an infinitesimal-area technique and then solved numerically to calculate the objective function and gradient vector. Minimization is done using a quasi-Newton algorithm that incorporates an active set method to enforce design constraints. This methodology is demonstrated by finding the optimal transient heater settings of a two-dimensional annealing furnace.

INTRODUCTION

Radiant enclosures are found in many industrial settings and are often used to heat a product uniformly according to a desired temperature history. Examples include annealing furnaces used in foundries, baking ovens that cook food, infrared heating systems that cure painted surfaces, and rapid thermal processing (RTP) chambers used to manufacture semiconductor wafers. In each case, the enclosure consists of a heater surface, several intermediate surfaces, and a design surface that contains the product to be processed. To design these systems, it is important to have an accurate model of the transient heater settings to size the heaters and other components of the enclosure. Having an accurate a priori estimate of the heater settings also enables the use of high-gain controllers that quickly adjust the heaters to compensate for deviations from the desired temperature distribution over the design surface.

In the past, transient heater settings have been solved using a forward "trial-and-error" design methodology, in which the designer first guesses the appropriate
heater settings and then repeatedly solves for the design surface temperature history, heuristically adjusting the heater settings until a satisfactory solution to the design problem is identified. The complicated nature of the coupled heat transfer modes makes an intuitive understanding of the system physics elusive, however, this technique usually requires many iterations and produces a final solution of limited quality.

To overcome this difficulty, designers have adapted model-based control algorithms to design the transient heater settings. At any instant, the difference between the temperature measured at different locations on the design surface and
the desired set-point temperature defines an error signal. This error signal is passed through a feedback loop to the controller, which, in turn, adjusts the heater settings to reduce the error signal. Model-based controllers have been applied to design the heaters in RTP furnaces [1, 2] and furnaces used in continuous annealing processes [3]. Despite the nonlinear nature of the problem, most model-based controllers use linear feedback algorithms that cannot accommodate the integrals in the radiosity equation that represent reflection and reradiation, so these effects are usually ignored. This induces large modeling errors into the controller, which severely limits its accuracy. Gwak and Masada [4] account for these effects by applying nonlinear control laws coupled with embedded Tikhonov and truncated singular value decomposition (TSVD) regularization.

More recently, inverse design methodologies have been developed to solve this type of design problem. In this approach, both the desired design surface temperature and the radiation heat input required to warm the product are specified throughout the process. The nonlinear system of equations is then linearized and written at discrete time steps throughout the process. The resulting set of ill-conditioned matrix equations is solved sequentially using a regularization method, starting from the first time step; each solution is then used to define the right-hand vector of the subsequent time step. Ertürk et al. [5] were the first to solve this design problem in this way, and used conjugate gradient regularization to solve the ill-conditioned system of equations at each time step. França and Howell [6] later developed a similar method based on TSVD regularization.

Unlike most model-based control algorithms, the inverse design approach can accommodate a sophisticated heat transfer model, which limits the modeling error. Also, by varying the degree of regularization used to solve the ill-conditioned equations, the designer can often generate a set of alternative solutions to the design problem. Nevertheless, a significant drawback of this method is that it is difficult to impose design constraints, and because of this, solutions from regularization often include regions of negative heat flux over the heater surface. This condition cannot be realized in practical furnaces, so these regions are usually taken to be adiabatic, impairing the solution quality.

Optimization through nonlinear programming overcomes many drawbacks of the above methods. In this approach, an objective function, \( F(\Phi) \), is defined in such a way that its minimum corresponds to the desired design outcome, which, in this case, is a temperature distribution over the design surface that both matches the desired temperature history and is also spatially uniform over the product throughout the process. The design parameters contained in \( \Phi \) define a set of functions that govern the heater output at any given process time. Gradient-based minimization algorithms are then employed to find the set of design parameters, \( \Phi^* \), that minimize the objective function, so that \( F(\Phi^*) = \text{Min}[F(\Phi)] \). The design parameters contained in \( \Phi^* \) correspond to the transient heater settings that produce a temperature distribution over the design surface that most closely satisfies the design requirements.

Since the design parameters are modified in an intelligent way at each iteration based on the local objective function curvature, the optimization design methodology requires fewer iterations than the forward design methodology, and the solution quality is much better. This technique can also accommodate a more sophisticated system model than most control algorithms and consequently is less susceptible to
modeling errors. Finally, unlike the inverse design methodology, the optimization methodology can easily accommodate design constraints. These may be used to force the calculated heat flux over the heater surface to lie within a specified operating range throughout the process.

Optimization has been applied on a limited basis to design industrial heating processes involving radiant enclosures. For example, unconstrained linear programming has been used to obtain the optimal heater settings for a simplified linearized model of an RTP furnace [7], and to obtain an initial estimate of the heater settings so that a high-gain controller could be used to operate an RTP furnace [8]. Nonlinear programming has also been used to optimize the heater settings for a continuous roll-through industrial furnace operating under steady-state conditions [9].

This article presents an optimization technique for designing the transient heater settings in manufacturing processes involving radiant heating, which accounts for sensible energy storage in the enclosure walls, as well as conjugate conduction and convection effects. The transient heater settings are optimized using a quasi-Newton minimization algorithm that incorporates an active set method to enforce the design constraints.

OPTIMIZATION STRATEGY

Figure 1 shows an example radiant enclosure. The design surface is located on the bottom surface and is irradiated by heaters on the top, which, in turn, are controlled throughout the process by design parameters contained in \( \Phi \). The underside of the design surface is insulated to prevent the energy provided by the heaters from leaving the system. As shown later, solving this problem requires discretization of both the temporal and spatial domains; the time domain is split into \( N_t \) time steps, the \( m \)th time step having a duration \( \Delta t_m \) starting from time \( t_{m-1} \) and ending at \( t_m \), and the design surface is split into \( N_{DS} \) discrete elements, with the \( i \)th element having an area \( \Delta A_i \). Furthermore, it is assumed that the density, \( \rho_{DS} \), specific heat, \( c_{DS} \), and thickness, \( D_{DS} \), are uniform over the design surface.

The goal of the optimization process is to identify the transient heater settings that produce an average design surface temperature that matches the desired set point temperature, \( T_{\text{target}}(t) \), at any instant, while simultaneously maintaining a uniform temperature over the design surface throughout the process. These heater settings correspond to the design parameters contained in \( \Phi^* \) that minimize the objective function

\[
F(\Phi) = \frac{1}{N_t N_{DS}} \sum_{m=1}^{N_t} \sum_{j=1}^{N_{DS}} [T_j(\Phi, t_m) - T_{\text{target}}(t_m)]^2
\]  

Since \( F(\Phi) \) is continuously differentiable, \( \Phi^* \) is found iteratively using gradient-based minimization. At the \( k \)th iteration, the design parameters are updated by first choosing a search direction, \( p^k \), based on the objective function topography. Next, a step size, \( \alpha^k \), is chosen, often by minimizing \( F(\Phi^k + \alpha^k p^k) \). Providing that the constraints are not violated, the design parameters are then updated by taking a “step” in the \( p^k \) direction, \( \Phi^{k+1} = \Phi^k + \alpha^k p^k \).
Newton’s method usually requires the fewest iterations to minimize the objective function; at each iteration, $p^k$ is found by solving

$$\nabla^2 F(\Phi^k) p^k = -\nabla F(\Phi^k)$$  \hspace{1cm} (2)$$

If the second-order design sensitivities contained in $\nabla^2 F(\Phi)$ are expensive to calculate, however, the quasi-Newton method is often more suitable. In this scheme, the search direction is given by

$$B^k p^k = -\nabla F(\Phi^k)$$  \hspace{1cm} (3)$$

where $B^k$ approximates $\nabla^2 F(\Phi^k)$. Initially, $B^0$ is equal to the identity matrix, and $p^0$ is the steepest-descent direction. In subsequent iterations, the Hessian approximation is updated and improved using values of $F(\Phi)$ and $\nabla F(\Phi)$ from previous iterations. The most popular way of doing this is with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update,

$$B^{k+1} = B^k - \frac{B^k s^k s^k B^k}{s^k B^k s^k} + \frac{y^k y^k}{y^k B^k p^k}$$  \hspace{1cm} (4)$$

with $s^k = \Phi^{k+1} - \Phi^k$ and $y^k = \nabla F(\Phi^{k+1}) - \nabla F(\Phi^k)$.
Since $B^k$ approximates $\nabla^2 F(\Phi^k)$ accurately only after several iterations, the quasi-Newton method typically requires more iterations than Newton's method to find $\Phi^*$. Nevertheless, the quasi-Newton method is usually more computationally efficient in cases where the second-order objective sensitivities are expensive to calculate.

In order to find $p^k$, it is necessary to evaluate $F(\Phi)$ and $\nabla F(\Phi)$, which, in turn, are calculated using temperatures, $T(\Phi, t_m)$, and first-order temperature sensitivities, $\partial T(\Phi, t_m)/\partial \Phi_p$, evaluated at discrete locations over the design surface throughout the process. A technique for doing this is presented in the next section.

**CALCULATION OF TEMPERATURE AND TEMPERATURE SENSITIVITIES**

An infinitesimal-area analysis [10] is used to derive the equations governing the temperature and temperature sensitivities. The first step of the analysis is to identify a suitable parametric representation for the enclosure. The enclosure geometry is specified by

$$ r = C(u) = [P(u), Q(u)]^T \quad a \leq u \leq b $$

where $C(u)$ is a vector function having components $P(u)$ and $Q(u)$ in the $x$ and $y$ directions, respectively. Thus, as $u$ varies over its domain, the position vector $r$ carves out the enclosure cross section in the $x-y$ plane.

Once the geometry is parameterized, either the temperature, $T(u, \Phi, t)$, or the heat flux, $q_s(u, \Phi, t)$, is specified at every location over the enclosure surface. In particular, the transient heat flux distribution over the heater surface is specified as a function of the design parameters contained in $\Phi$, and the adiabatic boundary condition is enforced over the design surface. Parametric functions representing emissivity, density, thermal conductivity, specific heat, and wall thickness are also defined over the domain of $u$ as $\varepsilon(u)$, $\rho(u)$, $\kappa(u)$, $c(u)$, and $\delta(u)$, respectively.

Once the enclosure is represented parametrically, the equation relating the radiosity distribution, $q_o(u, \Phi, t)$, to the temperature distribution, $T(u, \Phi, t)$, is derived by performing an energy balance on an infinitely-long wall element having a thickness $\delta(u)$ and an infinitesimal chord length $J(u) du$, as shown in Figure 2, where the surface discriminant, $J(u)$, is given by

$$ J(u) = \left\{ \frac{\partial P(u)}{\partial u} \right\}^2 + \left\{ \frac{\partial Q(u)}{\partial u} \right\}^2 \right\}^{1/2} $$

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In addition to radiation heat transfer, three other modes of heat transfer enter or leave the wall element: $q_{\text{cond}}(u, \Phi, t)$ is the net rate of heat transfer entering the wall element by conduction from the adjacent wall elements,

$$ q_{\text{cond}}(u, \Phi, t) = \frac{1}{J(u) \frac{\partial}{\partial u}} \left[ \frac{\kappa(u) \delta(u) \frac{\partial T(u)}{\partial u}}{J(u) \frac{\partial}{\partial u}} \right] $$
$q_{\text{conv}}(u, \Phi, t)$ is the rate of heat transferred by convection from the wall element to the fluid contained within the enclosure,

$$q_{\text{conv}}(u, \Phi, t) = h(u, t)[T(u, \Phi, t) - T_\infty(t)]$$  \hspace{1cm} (8)

and $q_s(u, \Phi, t)$ is the rate that heat enters the wall element by any other means (for example energy input from the heaters.) All of these terms are expressed per unit internal area of the infinitesimal wall element. (Convection heat transfer between the wall element and the fluid surrounding the enclosure has been excluded to simplify the heat transfer model, but could be added if this effect is significant.)

Setting the net rate of heat flux by radiation, external sources, conduction, and convection entering the infinitesimal wall element equal to the net rate of sensible energy per unit area, we find

$$-q_o(u, \Phi, t) + \int_a^b q_o(u', \Phi, t)k(u, u')du' + q_s(u, \Phi, t)$$

$$+ q_{\text{cond}}(u, \Phi, t) - q_{\text{conv}}(u, \Phi, t) = \rho(u)c(u)\delta(u)\frac{\partial T(u, \Phi, t)}{\partial t}$$  \hspace{1cm} (9)

where $k(u, u')$ contains geometric terms derived from Eq. (5). Equation (9) cannot be solved directly, however, since both the temperature and radiosity distributions are unknown. A more convenient form is found by performing another energy balance...
on the infinitesimal wall element at \( u' \); this results in an integro-differential equation, in which temperature is the only unknown variable,

\[
\sigma T^4(u, \Phi, t) - \int_a^b \sigma T^4(u', \Phi, t) k(u, u') du' = \frac{b(u', \Phi, t)}{\varepsilon(u)} - \int_a^b \frac{1 - \varepsilon(u')}{\varepsilon(u')} b(u', \Phi, t) k(u, u') du'
\]

where \( b(u, \Phi, t) \) represents the difference between the net nonradiative heat transfer into an infinitesimal wall element and sensible energy stored within that element,

\[
b(u, \Phi, t) = q_s(u, \Phi, t) + q_{\text{cond}}(u, \Phi, t) - q_{\text{conv}}(u, \Phi, t) - \rho(u) c(u) \delta(u) \frac{\partial T(u, \Phi, t)}{\partial t}
\]

The equations governing the temperature sensitivity are found by differentiating Eqs. (10) and (11), with respect to the design parameter of interest. By applying Liebnitz’s rule to Eq. (10) and noting that the integral bounds are independent of \( u \), the temperature sensitivities are governed by

\[
4\sigma T^3(u, \Phi, t) \frac{\partial T(u, \Phi, t)}{\partial \Phi_p} - \int_a^b 4\sigma T^3(u', \Phi, t) \frac{\partial T(u', \Phi, t)}{\partial \Phi_p} k(u, u') du' = \frac{1}{\varepsilon(u)} \frac{\partial b(u, \Phi, t)}{\partial \Phi_p} - \int_a^b \frac{1 - \varepsilon(u')}{\varepsilon(u')} \frac{\partial b(u', \Phi, t)}{\partial \Phi_p} k(u, u') du'
\]

where

\[
\frac{\partial b(u, \Phi, t)}{\partial \Phi_p} = \frac{\partial q_s(u, \Phi, t)}{\partial \Phi_p} - h(u, t) \frac{\partial T(u, \Phi, t)}{\partial \Phi_p} + \frac{1}{J(u)} \frac{\partial}{\partial u} \left[ \kappa(u) \delta(u) \frac{\partial^2 T(u, \Phi, t)}{\partial \Phi_p} \right]
\]

Since analytical solutions to integro-differential equations are usually not tractable, the temperature and temperature sensitivity distributions must be solved numerically. The parametric domain is first divided into \( N \) elements, with the \( i \)th element centered at \( u_i \) and having a width \( \Delta u_i \) in parametric space. Each of these elements corresponds to an infinitely long wall element having a finite width, as shown in Figure 3. The time domain is also discretized into \( N_t \) time steps, starting from \( t_0 \) and ending at \( t_{NT} = t_f \) in intervals of \( \Delta t_m \).

The integrals in Eqs. (10) and (12) are first approximated as discrete summations

\[
\int_a^b x(u') k(u, u') du' \approx \sum_{j=1}^{N} x(u_j) dF_{i-j\text{-strip}}
\]
where $dF_{i\rightarrow \text{strip}j}$ is the configuration factor between the point on the enclosure surface at $u_i$ and the exposed surface of a finite wall elements centered at $u_j$. Next, the temporal derivatives in Eq. (13) are approximated using a first-order backwards difference scheme,

$$\frac{\partial T(u, \Phi, t_m)}{\partial t} \approx \frac{T(u, \Phi, t_m) - T(u, \Phi, t_m-1)}{\Delta t_m}$$

(15)

Finally, the spatial derivatives in Eqs. (7) and (13) are rewritten using a second-order central difference approximation; if $\kappa(u)$ and $J(u)$ are uniform over each enclosure surface (which is often the case), then

$$q_{\text{cond}}(u_i, \Phi, t_m) \approx \frac{\kappa(u_i) \delta(u_i)}{J(u_i)^{1/2}} \left[ \frac{T(u_{i+1}, \Phi, t_m) - 2T(u_i, \Phi, t_m) + T(u_{i-1}, \Phi, t_m)}{\Delta t_i} \right]$$

(16)

By following these steps, the integro-differential equations governing the temperature distribution can be rewritten in discrete form,

$$\hat{T}_i^4(\Phi, \tau_m) - \sum_{j \neq i}^{N} \hat{T}_j^4(\Phi, \tau_m) dF_{i\rightarrow \text{strip}j}$$

$$= \frac{\hat{b}_i(\Phi, \tau_m, \tau_{m-1})}{\varepsilon_i} - \sum_{j \neq i}^{N} \frac{1 - \varepsilon_j}{\varepsilon_j} \hat{b}_j(\Phi, \tau_m, \tau_{m-1}) dF_{i\rightarrow \text{strip}j}$$

(17)
where \( \varepsilon_t = \varepsilon(u_t) \), and the time and temperature terms are represented nondimensionally by \( \tau_m = (t_m - t_0) / (t_f - t_0) \) and \( \bar{T}_i(\Phi, \tau_m) = T(u_i, \Phi, t_m) / T_s \), where \( T_s \) is a scaling temperature. Also,

\[
\dot{b}_i(\Phi, \tau_t, \tau_{m-1}) = \dot{q}_{si}(\Phi, \tau_m) + C_{\text{cond}} \frac{\hat{k}_i \delta_i}{\bar{T}_i^2} \left[ \frac{\dot{\bar{T}}_{i+1}(\Phi, \tau_m) - 2 \bar{T}_i(\Phi, \tau_m) + \dot{\bar{T}}_{i-1}(\Phi, \tau_m)}{\Delta \tau_i^2} \right] \\
+ C_{\text{conv}} \dot{h}(\tau_m) \left[ \bar{T}_i(\Phi, \tau_m) - \bar{T}_\infty(\tau_m) \right] \\
+ C_{\text{trans}} \rho_i c_i \delta_i \frac{\bar{T}_i(\Phi, \tau_m) - \bar{T}_i(\Phi, \tau_{m-1})}{\Delta \tau_m}
\]

(18)

where \( \dot{q}_{si}(\Phi, \tau_m) = q_{si}(u_i, \Phi, t_m) / \sigma \bar{T}_i^4 \), and \( \dot{h}(\tau_m) = h(t_m) / \bar{h} \). The enclosure properties are represented by \( \bar{k}_i = k(u_i) / \bar{k} \), \( \bar{\rho}_i = \rho(u_i) / \bar{\rho} \), \( \bar{c}_i = c(u_i) / \bar{c} \), \( \delta_i = \delta(u_i) / \delta_0 \), and \( \bar{J}_i = J(u_i) / L_s \), where \( L_s \) is a characteristic length of an enclosure surface. The coefficients \( C_{\text{cond}}, C_{\text{conv}}, \) and \( C_{\text{trans}} \) are given by

\[
C_{\text{cond}} = \frac{\bar{k}_i \delta_i}{\sigma \bar{T}_i^3 L_c^2}
\]

(19)

\[
C_{\text{conv}} = \frac{\bar{h}}{\sigma \bar{T}_i^3}
\]

(20)

and

\[
C_{\text{trans}} = \frac{\rho_i c_i \delta_i}{\sigma \bar{T}_i^3 (t_f - t_0)}
\]

(21)

and their relative magnitudes indicate the importance of conduction, convection, and sensible energy storage compared to radiation heat transfer.

Writing Eqs. (17) and (18) for every wall element results in a matrix equation governing the temperature of the enclosure at time \( \tau_m \),

\[
A_1 \dot{T}_i(\Phi, \tau_m) + A_2 \bar{T}(\Phi, \tau_m) = c(\Phi, \tau_m, \tau_{m-1})
\]

(22)

where \( \dot{T}_i(\Phi, \tau_m) = \{ T_1^i, T_2^i, \ldots, T_n^i \}^T \) and \( \bar{T}(\Phi, \tau_m) = \{ \bar{T}_1, \bar{T}_2, \ldots, \bar{T}_n \}^T \). In order to solve for the transient temperature distribution, Eq. (22) must be linearized to form a related matrix equation,

\[
A(\Phi, \tau_{m-1}) x(\Phi, \tau_m) = b(\Phi, \tau_m, \tau_{m-1})
\]

(23)

where \( x_i(\Phi, \tau_m) = T_i(\Phi, \tau_m) \) or \( \bar{T}_i(\Phi, \tau_m) \), depending on what linearization scheme is used. Although the relative magnitudes of \( C_{\text{cond}}, C_{\text{conv}}, \) and \( C_{\text{trans}} \) provide some guidance when choosing a linearization scheme, it is also necessary to consider the smallness of the \( \Delta \tau \) and \( \Delta \mu \) values needed to obtain a grid-independent solution to a specific problem. If a large number of wall elements is required to obtain grid independence, for example, then \( \Delta \mu \) is very small, and the coefficients contained in \( A_2 \)
tend to be larger in magnitude than those in $A_1$. In this scenario, Eq. (22) should most likely be linearized by lagging the radiation terms in order to obtain a convergent solution. The transient temperature distribution is then solved by first guessing a temperature distribution at $\tau = 0$ and then writing and solving Eq. (23) at each time step using the temperature distribution from the previous time step to form the $A$ matrix and $b$ vector of the current time step.

Applying a similar procedure to Eqs. (12) and (13) results in another matrix equation for the temperature sensitivities with respect to each design parameter,

$$dA(\Phi, \tau_{m-1})x'(\Phi, \tau_m) = b'(\Phi, \tau_m, \tau_{m-1}) \quad (24)$$

where $x'_i(\Phi, \tau_m) = \partial T_i(\Phi, \tau_m) / \partial \Phi_p$. As with the temperature distribution, the sensitivities are found by guessing a sensitivity distribution at $\tau_0$ and then writing and solving Eq. (24) at each time step using the sensitivities from the previous time step.

**IMPLEMENTATION**

The design methodology described in the previous section is demonstrated by using it to optimize the heater settings of the two-dimensional annealing furnace shown in Figure 4. The top surface is composed of 10 uniformly spaced heaters, the two side walls are refractory surfaces, and the design surface is located on the bottom of the enclosure. The heater and refractory surfaces have the properties of refractory brick, and the design surface is a sheet of AISI 1010 steel. All enclosure surfaces are assumed to be gray and diffuse, and their properties are summarized in Table 1.

The objective of the design problem is to determine the transient heater settings that heat the steel at a linear ramp rate from 300 to 500 K over a 5-h period. It is assumed that the enclosure surfaces start at thermal equilibrium at 300 K, at which point the heaters are activated, and the surfaces are exposed to a fluid at $T_\infty = 500$ K and $h = 5$ W/m² K. The enclosure surfaces are assumed to be thermally isolated from each other, which is enforced by insulating the surface edges.

![Figure 4. Example design problem. (Heater numbers are shown in circles.)](image-url)
The heat flux distribution over each heater is assumed to be uniform and is controlled by a cubic spline function of nondimensional time. Because of symmetry, the heaters are controlled in pairs and are numbered as shown in Figure 4. In particular, if \( u_i \) lies on the \( h \)th heater,

\[
q_{hi}(\Phi, \tau) = \Phi_{4h-3}(1 - \tau)^3 + \Phi_{4h-2}3(1 - \tau)^2\tau + \Phi_{4h-1}3(1 - \tau)\tau^2 + \Phi_{4h}\tau^3
\]

where \( \{\Phi_{4h-3}, \Phi_{4h-2}, \Phi_{4h-1}, \Phi_{4h}\}^T \) is a subvector of \( \Phi \); thus, 20 design parameters specify the heat flux distribution over the heater surface throughout the process. Controlling the heater output this way reduces the dimension of the optimization problem and acts to regularize the solution, since cubic splines are smooth functions. Also, because the basis functions in Eq. (25) sum to unity for any value of \( \tau \), the output of each heater can be constrained to lie between upper and lower bounds by applying these same bounds to the corresponding design parameters. In this example, the heater outputs are constrained to lie between 0 \( /C_{20}^q \) and 10 by incorporating an active set method [12] into the BFGS minimization routine.

The problem was nondimensionalized using \( L_c = 1 \) m, \( h = 5 \) W/m\(^2\) K, and \( T_s = 1,000 \) K, while \( \kappa_s, \rho_s, \epsilon_s, \) and \( \delta_s \) were set equal to the corresponding design surface properties. Substituting these values into Eqs. (19)–(21) results in \( C_{\text{cond}} = 0.0225, C_{\text{conv}} = 0.0882, \) and \( C_{\text{trans}} = 0.2488. \) After several attempts, a convergent solution was found by lagging the emissive power terms, resulting in a matrix equation having the form of Eq. (23), where \( A(\Phi, \tau_m) \) contains the conduction, convection, and sensible energy temperature coefficients, \( b(\Phi, \tau_m, \tau_{m-1}) \) is composed of the heat fluxes, conduction boundary condition, fluid temperature, and sensible energy and thermal radiation terms from the previous time step, and \( x_i(\Phi, \tau_m) = \bar{T}_i(\Phi, \tau_m). \) Solving for the sensitivities results in

\[
A(\Phi, \tau_m)x'(\Phi, \tau_m) = b'(\Phi, \tau_m, \tau_{m-1})
\]

where \( b'(\Phi, \tau_m, \tau_{m-1}) \) contains the heat flux, sensible energy, and thermal radiation sensitivities with respect to \( \Phi_p, \) and \( x_i'(\Phi, \tau_m) = \partial \bar{T}_i(\Phi, \tau_m)/\partial \Phi_p. \) Thus, \( A(\Phi, \tau_m) \) needs to be formed and inverted only once at each time step in order to calculate both the temperature and temperature sensitivities.

The optimal heater settings are found by minimizing the objective function defined in Eq. (1). The parametric and temporal domains were discretized into \( N = 240 \) wall elements and \( N_t = 500 \) time steps to calculate \( f(\Phi) \) and \( \nabla f(\Phi) \)

<table>
<thead>
<tr>
<th>Table 1. Enclosure surface properties</th>
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<td>( \kappa ) (W/m K)</td>
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</table>
throughout the optimization process. The minimization was carried out starting from $F_0^i = 1, i = 1, 2, \ldots, 20$, and was stopped when $\|V_{FR}(\Phi^*)\| < 10^{-6}$, where $V_{FR}(\Phi^*)$ contains the first-order objective function sensitivities with respect to the design parameters that are not held at an active constraint. It was necessary to periodically stabilize the quasi-Newton algorithm throughout the minimization process by setting $B(\Phi^*)$ equal to the identity matrix whenever its condition number exceeded 100.

A local minimum of $F(\Phi^*) = 3.057 \times 10^{-4}$ was found after 66 iterations. The optimal set of design parameters is shown in Table 2, and the corresponding heater settings are shown in Figure 5. The resulting transient temperature response over the design surface is shown in Figure 6. The average design surface temperature was below the set-point temperature from $t = 0$ to $t = 0.764$ and was slightly above the set-point temperature throughout the remainder of the process. The maximum deviation of the average design surface temperature from the set-point temperature was $-7.79\%$, occurring at $t = 0.236$, and the final temperature exceeded the set-point temperature by $6.07\%$ at the end of the process. While a better solution might be found by using higher-order splines to control the heater settings, the sensitivity of the design surface temperature to the heater inputs is limited by the thermal inertia of the enclosure walls. (A better result would be obtained by preheating the oven instead of starting the process from thermal equilibrium at 300 K.) A near-uniform temperature distribution was maintained throughout the process; the maximum standard deviation of the design surface temperature was less than $0.35\%$, occurring at $t = 1$.

Due to the approximations made in Eqs. (14)–(16), it is necessary to perform both spatial and temporal refinement studies to assert the grid independence of the optimal solution, which are shown in Figures 7 and 8, respectively. (Log-log plots were not used, since the magnitudes of the discretization errors are important rather than the order of convergence.) These studies are based on comparing the temperatures calculated at the center of the plate at $t = 0.5$ and $t = 1$ with the corresponding grid-independent values, which are taken to be the temperature distributions obtained using $N = 480$ and $N_t = 500$ for the spatial refinement study, and $N = 240$ and $N_t = 1,000$ for the temporal refinement study. Both studies show that choosing $N = 240$ and $N_t = 500$ provides a sufficient level of grid refinement. Grid independence can also be verified by demonstrating energy conservation, which is quantified by the energy imbalance,

$$\%\text{EI}(\Phi, N, N_t) = \frac{Q_{\text{in}}(\Phi) + Q_{\text{conv}}(\Phi, N, N_t) - Q_{\text{stored}}(\Phi, N, N_t)}{|Q_{\text{in}}(\Phi)| + |Q_{\text{conv}}(\Phi, N, N_t)| + |Q_{\text{stored}}(\Phi, N, N_t)|} \times 100\%$$

Table 2. Optimal design parameters

<table>
<thead>
<tr>
<th>Heater 1</th>
<th>Heater 2</th>
<th>Heater 3</th>
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<td>0.2569</td>
<td>0.2429</td>
<td>0.2349</td>
</tr>
</tbody>
</table>
Figure 5. Optimal heater settings.

Figure 6. Optimal design surface temperature.
Figure 7. Spatial grid refinement and energy balance study. ($N_t = 500$.)

Figure 8. Temporal grid refinement study. ($N = 240$.)
where $Q_{\text{in}}(\Phi)$ is the energy added to the system by the heaters, $Q_{\text{conv}}(\Phi, N, N_t)$ is the energy transferred to the wall elements from the fluid, and $Q_{\text{stored}}(\Phi, N, N_t)$ is the sensible energy stored by the wall elements throughout the process. Although $\%\text{EI}$ should equal zero for an exact solution, the steps performed when discretizing the governing equations [principally Eq. (14)] mean that energy conservation is not exactly satisfied. Figure 7 shows that this error becomes very small, however, as $N$ becomes large.

It should be noted that this solution corresponds to one local minimum in the domain of $\Phi$. There may, in fact, be many local minima (including an unknown global minimum). If this is the case, the quasi-Newton algorithm could be run from different initial points in order to find a different local minimum having a smaller value of $F(\Phi^*)$. Alternatively, the minimization could be carried out using a more sophisticated (but more expensive) algorithm designed specially for minimizing multimodal objective functions, like the one presented in [13].

**CONCLUSIONS AND FUTURE WORK**

This article has presented an optimization method for finding the optimal heater settings in radiant enclosures used in manufacturing processes. In this approach, the design problem is transformed into a multivariate minimization problem by defining an objective function that is minimized when the average design surface temperature matches the set-point temperature and the temperature distribution is uniform over the design surface throughout the process. Once this is done, the heater settings are optimized by minimizing the objective function with a BFGS quasi-Newton method that incorporates an active set method to impose bounds on the heater outputs.

Because the design is improved systematically at each iteration, this technique requires far less design time and provides a much better final solution than the traditional “trial-and-error” approach, which, instead, relies solely on the designer’s intuition and experience to solve the problem. This method can also accommodate a sophisticated heat transfer model and design constraints that help ensure that the optimal solution can be implemented in a practical setting.

The authors intend to refine this method further by using a higher-order function to control the transient heater output and by adopting a method based on exchange factors for analyzing the radiant heat transfer, which will enable the treatment of enclosures containing surfaces with directionally dependent optical properties.

**REFERENCES**


