Markov Chain Problems
Solutions

1. a) States: No of machines requiring service
   Transitions: Passage of 5 minutes

   \[
   P = \begin{bmatrix}
   .36 & .48 & .16 \\
   .6 & .4 & 1 \\
   0 & 1 & 0 \\
   \end{bmatrix}
   \]

   \( \pi^{(1)} = [.36 \  .48 \  .16] \)

   b) \( \pi^{(2)} = [.4176 \  .5248 \  .0576] \)

   c) \( \pi = [.4491 \  .4790 \  .0719] \)

   \( P(\text{idle}) = \pi_1 = .4491 \)

   \( \text{Fraction (busy)} = 1 - \pi_1 = .5509 \)

   \( E[\text{no. requiring service}] = 0 \times .4491 + 1 \times .4790 + 2 \times .0719 = .6228 \)

   d) \( E[\text{cost}] = 96 \times .6228 \times $5 = $298.94 \)

2. a) States: # weeks a person has the disease
   Transitions: Passage of weeks

   \[
   P_{\text{no drugs}} = \begin{bmatrix}
   .9 & .1 & 0 & 0 \\
   0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 1 \\
   1 & 0 & 0 & 0 \\
   \end{bmatrix}
   \]

   \[
   \pi_{\text{no drugs}} = \begin{bmatrix}
   0.769230769 \\
   0.076923077 \\
   0.076923077 \\
   0.076923077 \\
   \end{bmatrix}
   \]

   b) \( \text{Fraction ill: } \pi_1 + \pi_2 + \pi_3 \text{ No Drugs: 0.23077} \)
\[
\begin{align*}
\text{c) } & \quad P_{\text{with drugs}} = \begin{bmatrix}
.9 & .1 & 0 & 0 \\
.5 & 0 & .5 & 0 \\
.5 & 0 & 0 & .5 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad \pi_{\text{drugs}} = \begin{bmatrix}
0.85106 \\
0.08511 \\
0.04255 \\
0.02128
\end{bmatrix} \\

\text{Fraction of infected population ill shorter time: } & \quad \frac{\pi_1 + \pi_2}{\pi_1 + \pi_2 + \pi_3} = 0.86 \\
\text{d) } & \quad \text{Cost no drugs: } 50 \times [\pi_1 + \pi_2 + \pi_3] = 11.54 \\
& \quad \text{Cost with drugs: } 5 + 50 \times [\pi_1 + \pi_2 + \pi_3] = 12.45 \\
& \quad \text{So, it is not cost effective to use the drugs.}
\end{align*}
\]

3. a) States: Location
   Transitions: Trips
   \[
P = \begin{bmatrix}
.6 & .4 \\
.25 & .75
\end{bmatrix}
\]

b) \( p_{11}^{(4)} = 0.394 \), \( p_{12}^{(4)} = 0.606 \)

c) \( \pi = [0.3846 \ 0.6154] \)
   \( E[\text{no trips } i \to j] = \pi_i p_{ij} \)
   \[
   \begin{bmatrix}
.231 & .154 \\
.154 & .461
\end{bmatrix}
\]

d) \( \text{Intake} = \sum s_{ij} \times E[\text{no trips } i \to j] = 5.16 \)

e) Change transitions matrix for each of the 4 scenarios. and repeat c) The optimal policy is the one that yields the greatest revenue. It is to cruise SF and go to the bus terminal in OA.

\[
P = \begin{bmatrix}
.6 & .4 \\
.4 & .6
\end{bmatrix}
\]

\[
\pi = [0.5 \ \ 0.5]
\]

\( \text{Intake} = \sum s_{ij} \times E[\text{no trips } i \to j] = 5.80 \)
4. a) States: Price of Stock  
Transitions: Passage of one day  
\[
P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .2 & .3 & .5 & 0 \\ 0 & .2 & .3 & .5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]  
37 and 40 are absorbing states, 38 and 39 are transient states  

b)  
\[
\begin{array}{ccccc} 
  & n & p_{37}^{(n)} & p_{38}^{(n)} & p_{39}^{(n)} & p_{40}^{(n)} \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & .2 & .3 & .5 & 0 & \\
 2 & .26 & .19 & .30 & .25 & \\
 3 & .298 & .117 & .185 & .400 & \\
 4 & .3214 & .0721 & .1140 & .4925 & \\
\end{array}
\]  
\[E[G\text{ain}] = -1[.3214] + 1[.1140] + 2[.4925] = .7776\]

c) Given the absorbing states 37 and 40, then the \( p_{ij}^{(n)} = f_{ij}^{(n)} \), so the expected time is easy to compute. The time until sale is the time that States 38 and 39 are occupied. Continuing the table above:  
\[
\begin{array}{ccccc} 
  & n & p_{37}^{(n)} & p_{38}^{(n)} & p_{39}^{(n)} & p_{40}^{(n)} \\
 5 & 0.33582 & 0.04443 & 0.07025 & 0.5495 & \\
 6 & 0.344706 & 0.027379 & 0.04329 & 0.584625 & \\
 7 & 0.350182 & 0.016872 & 0.026677 & 0.60627 & \\
 8 & 0.353556 & 0.010397 & 0.016439 & 0.619608 & \\
 9 & 0.355636 & 0.006407 & 0.01013 & 0.627828 & \\
 10 & 0.356917 & 0.003948 & 0.006242 & 0.632893 & \\
 11 & 0.357706 & 0.002433 & 0.003847 & 0.636014 & \\
 12 & 0.358193 & 0.001499 & 0.00237 & 0.637937 & \\
 13 & 0.358493 & 0.000924 & 0.001461 & 0.639122 & \\
 14 & 0.358678 & 0.000569 & 0.0009 & 0.639853 & \\
\end{array}
\]
So, \(E[T] = 1(.3+.5) + 2(.19+.30)....... = 5.27\) days  

5. a)  
1: All states recurrent, aperiodic, non-null states. Ergodic  
2: All states recurrent, aperiodic, non-null states. Ergodic.
Now, for this simple problem, you can visually see that the solution is 1/6 from the network since there is only one path that you can take. However, you should learn to use these relationships.

You can either solve for the steady state probabilities and the look at the inverse or solve directly for the steady state mean values:

\[
\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} \mu_{kj}
\]

\[
\mu_{13} = 1 + p_{11} \mu_{13} + p_{12} \mu_{23}
\]

\[
\mu_{23} = 1 + p_{21} \mu_{13} + p_{22} \mu_{23}
\]

\[
\mu_{33} = 1 + p_{31} \mu_{13} + p_{32} \mu_{23}
\]

\[
\mu_{13} = 5, \mu_{23} = 7/2, \mu_{33} = 15/8
\]

6 a) This problem is one involving probabilities of first return, inherently, since you cannot leave an absorbing state. Recall the form of the problem that we used for transient sets of states and absorbing sets of states:

\[
Q = \begin{bmatrix}
0 & .6 & 0 & 0 \\
0 & 0 & .5 & 0 \\
0 & 0 & 0 & .4 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
R = \begin{bmatrix}
.4 & 0 \\
.5 & 0 \\
.6 & 0 \\
.7 & .3
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
I & 0 \\
R & Q
\end{bmatrix}
\]

For this problem, the probabilities of the absorbing states are:

\[
A = [I - Q]^{-1} R
\]

\[
[I - Q]^{-1} = \begin{bmatrix}
1 & .6 & .3 & .12 \\
0 & 1 & .5 & .2 \\
0 & 0 & 1 & .4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
A = [I - Q]^{-1} R = \begin{bmatrix}
.964 & .036 \\
.940 & .060 \\
.880 & .120 \\
.700 & .300
\end{bmatrix}
\]

(Rows are time I, columns are absorbing probabilities)
Time 1= New: So use $a_{11} = .964$

b) Time 2= One month overdue: So, use $a_{12} = .06$

c) From a, we can see that 3.6% of all new accounts go uncollected. With yearly accounts of $1,200,000$, $43,200 (0.036 \times 1.2$ Million) go uncollected.