

Part A	Experiments in Fluid Mechanics
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A1 The Experiment as a Boundary-Value Problem

A fluid flow experiment is an attempt to isolate a part of the world and measure flow and thermodynamic properties. A fluid is defined as a material that deforms continuously if a shear stress is applied. An internal flow situation has walls bounding the flow, but an inflow and outflow position must be controlled. An external flow problem has a uniform flow at far distances from the body of interest. In both situations the state of flow at the boundary is controlled. In the mathematical representation of the flow, the flow conditions on the boundary are specified. This is the nature of the governing physics. If the boundary conditions depend on time the flow situation in the entire region must be specified at the initial time.

In what follows the major physical laws are outlined. In most cases tensor calculus in symbolic form is employed. Scalars are lightface type, vectors are boldface type and tensors are boldface capitals. However, in cases where confusion is possible with tensor multiplications, index notation is employed. Scalars are then without an index, vectors have one index and tensors have two or more indices.

A1.1 Thermodynamic Equations

Continuum Properties

The properties of a continuum are defined by an imaginary experiment where a region of volume V with characteristic length L is imagined to contain molecules. At a given position the volume is reduced around that position as indicated by the limit process $L \rightarrow 0$. A typical molecule denoted by the subscript i has a mass m_i and an instantaneous velocity \mathbf{v}_i . The density is the sum of mass over all molecules in the region divided by the volume as the limit is taken. Although $L \rightarrow 0$ is indicated, it cannot become so small that fluctuations occur because only a few molecules are present.

$$\rho = \lim_{L \rightarrow 0} \frac{m_i}{V} \quad \text{A1.1}$$

The mass averaged velocity is a vector average of the molecular velocities and mass. This is appropriate to measure the momentum.

$$\mathbf{v} = \lim_{L \rightarrow 0} \frac{m_i \mathbf{v}_i}{m_i} \quad \text{A1.2}$$

If the substance has several chemical species, $n^{(k)}$ moles in the region, a molar-averaged velocity for each species k is

$$\mathbf{v}^{(k)} = \lim_{L \rightarrow 0} \frac{\mathbf{v}_i^{(k)}}{n^{(k)}} \quad \text{A1.3}$$

Such a velocity is useful in diffusion problems. The internal energy (per unit mass) due to random translational motions of the molecules is

$$e = \lim_{L \rightarrow 0} \frac{\frac{1}{2} m_i (\mathbf{v}_i - \mathbf{v}) \cdot (\mathbf{v}_i - \mathbf{v})}{m_i} \quad \text{A1.4}$$

The total internal energy includes other molecular motions such as vibrations, and configuration energies. The properties above are well defined whether or not the substance is in thermodynamic equilibrium.

Thermodynamics

It is assumed that the bulk motion of the substance does not affect the thermodynamic state. All thermodynamic variables of a simple compressible substance are described by a fundamental law that gives the entropy $s = s(\rho, e)$ or in another form $e = e(s, \rho)$. Each substance has its own entropy function, however, all functions obey the fundamental differential equation of thermodynamics.

$$e = e(s, \rho) \quad \text{A1.5}$$

$$d e = T d s - p d(\rho^{-1}) \quad \text{A1.6}$$

The thermodynamic pressure is defined by

$$p(s, \rho) = - \left. \frac{e}{(\rho^{-1})} \right|_s \quad \text{A1.7}$$

And the temperature is given by

$$T(s, \rho) = \left. \frac{e}{s} \right|_\rho \quad \text{A1.8}$$

Other thermodynamic properties follow from their definitions, for example the enthalpy $h = e + p / \rho$.

Two equations of state are equivalent to the fundamental law of a substance. The first equation of state is of the form

$$p = p(\rho, T) \quad \text{A1.9}$$

or

$$= (p, T) \quad \text{A1.10}$$

It is equivalent to specify the compressibility coefficient functions.

$$\alpha(p, T) = \frac{1}{\rho} \left. \frac{\partial \rho}{\partial p} \right|_T \quad \text{A1.11}$$

$$\beta(p, T) = -\frac{1}{\rho} \left. \frac{\partial \rho}{\partial T} \right|_p \quad \text{A1.12}$$

Integration these functions will reproduce $= (p, T)$.

The second equation of state is an energy equation of state.

$$e = e(\rho, T) \quad \text{A1.13}$$

The important derivative function here is the specific heat (per unit mass) at constant volume.

$$c_v(\rho, T) = \left. \frac{\partial e}{\partial T} \right|_\rho \quad \text{A1.14}$$

The other function $\left. \frac{\partial e}{\partial \rho} \right|_T$ is related to the state equation $= (p, T)$ by thermodynamic theory. In summary, the functions $p = p(\rho, T)$ relation and $c_v = c_v(\rho, T)$ describe the thermodynamics of a substance.

Often the enthalpy $h = e + p/\rho$ is used in preference to the internal energy. The important derivative function here is the specific heat (per unit mass) at constant pressure.

$$c_p(p, T) = \left. \frac{\partial h}{\partial T} \right|_p \quad \text{A1.14}$$

The other function $\left. \frac{\partial h}{\partial p} \right|_T$ is related to the state equation $= (p, T)$ by thermodynamic theory. Alternatively, the functions $p = p(\rho, T)$ relation and $c_p = c_p(p, T)$ describe the thermodynamics of a substance.

There are special approximations of importance; perfect gas, ideal gas, and incompressible fluid. For a perfect gas the state equations are:

$$p = \rho R T \quad \text{A1.15}$$

$$e = c_v(\rho, T) T \quad \text{A1.16}$$

Alternatively, $h = c_p(\rho, T) T$ can replace the energy equation. A further restriction to an ideal gas gives simpler forms,

$$e = c_v(T) T \quad \text{A1.17}$$

$$h = c_v(T) T + p / \rho \quad \text{A1.18}$$

$$c_p(T) = c_v(T) + R \quad \text{A1.19}$$

Here R is the specific gas constant.

An incompressible fluid has thermodynamic variables that are independent of the density. The fundamental equation is

$$s = s(e). \quad \text{A1.20}$$

$$d e = T d s \quad \text{A1.21}$$

As before, the temperature is defined by

$$T = \left. \frac{e}{s} \right|_{\rho} \quad \text{A1.22}$$

Pressure is a mechanical variable independent of the thermodynamic state.

The first equation of state does not exist, and the second equation of state is

$$e = e(T) = c_v(T) T \quad \text{A1.23}$$

The enthalpy is a mixture of thermodynamic and mechanical properties,

$$h = e + p / \rho.$$

A1.2 Kinematic Equations

A fluid particle is an imaginary collection of fluid that locally follows the fluid velocity. Due to random molecular motions a fluid particle does not consist of the same molecules for all time. There are two mathematical viewpoints with different independent space and time variables. The dependent variables, the thermodynamic properties and characteristics of the continuum motion (velocity vorticity, strain rate, etc.) are instantaneous concepts and the same from either viewpoint. The Lagrangian view can be thought of as a history of a certain fluid particle. Independent variables in the Lagrangian viewpoint are the initial position of the particle and the time, \mathbf{x}^0 , and \hat{t} . The particle position vector \mathbf{r} is

$$\mathbf{r} = \tilde{\mathbf{r}}(\mathbf{x}^0, \hat{t}) \quad \text{A1.24}$$

It follows from this that the particle velocity is

$$\mathbf{v} = \frac{\tilde{\mathbf{r}}}{\hat{t}} \quad \text{A1.25}$$

and the particle acceleration is

$$\mathbf{a} = \frac{\mathbf{v}}{\hat{t}} \quad \text{A1.26}$$

Alternately, the Eulerian viewpoint is a fixed position in space and time. The independent variables are \mathbf{x} and t . In this viewpoint the particle position is

$$\mathbf{r} = \mathbf{r}(\mathbf{x}, t) = \mathbf{x} \quad \text{A1.27}$$

Equating \mathbf{r} and time provides the connection between the two viewpoints.

$$\mathbf{x} = \tilde{\mathbf{r}}(\mathbf{x}^0, \hat{t}) \quad \text{A1.28}$$

$$t = \hat{t} \quad \text{A1.29}$$

For a given particle \mathbf{x}^0 , the particle path, with time as a parameter, is

$$\mathbf{x}_p = \tilde{\mathbf{r}}(\mathbf{x}^0, t) \quad \text{A1.30}$$

Choosing a different initial particle, \mathbf{x}^0 , gives a different particle path.

Next, consider a line of particles given by an equation $\mathbf{x}^0 = \mathbf{x}^0(a)$ where a is a parameter that varies over some range. For example, say $0 \leq a \leq 1$, $a = 0$ is the beginning of the line and $a = 1$ is the end of the line. For given time t , a streak line of particles originally at \mathbf{x}^0 are at

$$\mathbf{x}_{str} = \tilde{\mathbf{r}}(\mathbf{x}^0(a), t) \quad \text{A1.31}$$

A line of marked particles would move through the flow according to Eq.A1.31

.

Another important concept is that of a streamline. For given time t , a streamline is everywhere tangent to the velocity

$$\begin{aligned} \mathbf{v} \times d\mathbf{x}_{stm} &= 0 \\ \text{or} & \\ \frac{dx}{u} &= \frac{dy}{v} = \frac{dz}{w} \end{aligned} \quad \text{A1.32}$$

The second version above refers to a coordinate system x, y, z with velocity components u, v, w .

Any dependent property f can be expressed in Eulerian variables $f = f_E(\mathbf{x}, t)$ or in Lagrangian variables $f = f_L(\mathbf{x}^0, \hat{t})$. A Lagrangian time rate of change, the rate of change following a fluid particle is given by

$$\frac{f_L}{\hat{t}} \quad \text{A1.33}$$

Particle velocity and acceleration are

$$v_i = \frac{\tilde{r}_i}{\hat{t}} = \frac{D x_i}{D t} \quad \text{A1.34}$$

$$a_i = \frac{v_i}{\hat{t}} \quad \text{A1.35}$$

By using the chain rules of differentiation of a composite function one finds that the Eulerian representation of a Lagrangian time derivative is

$$\frac{f_L}{\hat{t}} = \frac{f_E}{t} + \mathbf{v} \cdot \nabla f_E \quad \text{A1.36}$$

The right hand side is called the Stokes derivative, substantial derivative, or material derivative.

$$\frac{D f_E}{D t} = \frac{f_E}{t} + \mathbf{v}_i \frac{f_E}{x_i}$$

This offers a physical interpretation to this combination of terms.

In addition to the translational velocity, every point in the fluid has a vorticity. Given a fluid particle P at \mathbf{x} , the solid-like rotation of a near neighbor at P' is the angular velocity $\boldsymbol{\omega}(\mathbf{x})/2$. By definition

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \quad \text{A1.37}$$

The rate of strain at P is defined as

$$S_{ij} = \frac{1}{2} (v_i v_j + v_j v_i) ; \quad \mathbf{S} = \frac{1}{2} \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \quad \text{A1.38}$$

Consider two particles P at \mathbf{x} and a near neighbor at P' at a distance ds in direction of a unit vector \mathbf{n} from P . The strain velocity per unit distance is the strain vector \mathbf{d} .

$$\frac{d \mathbf{v}_{\text{strain}}}{d s} = \mathbf{d} = \mathbf{n} \cdot \mathbf{S} \quad \text{A1.39}$$

A small material region has a volumetric rate of expansion

$$\lim_{L \rightarrow 0} \frac{1}{V_{MR}} \frac{D V_{MR}}{D t} = \mathbf{v} \cdot \mathbf{v} \quad \text{A1.40}$$

This is of course zero for an incompressible flow.

A1.3 Balance Laws and Local Governing Equations

Continuity

The law for conservation of mass yields the "continuity" equation. The latter name referring to the fact the underlying assumption of a continuum is required.

$$\frac{1}{\rho} \frac{D \rho}{D t} + \mathbf{v} \cdot \mathbf{v} = 0 \quad \text{A1.41}$$

The fractional rate of change of density following a fluid particle and the rate of expansion of the material region form a balance. Another viewpoint, from a fixed point in space, gives a balance of the local rate of change of density and the divergence of the flux of fluid into the point.

$$\frac{\rho}{t} + \mathbf{v} \cdot (\rho \mathbf{v}) = 0 \quad \text{A1.42}$$

The continuity equation is used to derive a relation between the substantial derivative of any fluid property f_E and the local and convective derivatives of f_E observed at a fixed Eulerian location.

$$\rho \frac{f_E}{t} + \mathbf{v} \cdot \nabla f_E = \frac{(\rho f_E)}{t} + \mathbf{v} \cdot (\nabla (\rho f_E)) \quad \text{A1.43}$$

Linear momentum and related equations

The linear momentum per unit mass, \mathbf{v} , responds to surface and volumetric forces. The surface stress \mathbf{R} on an area with outward normal \mathbf{n} is

the force per unit area of the outside substance upon the inside substance. The variation of this stress with surface direction is given by \mathbf{n} and the variation with location is given by a stress tensor $\mathbf{T}(\mathbf{x}, t)$.

$$\mathbf{R} = \mathbf{n} \cdot \mathbf{T} \quad \text{A1.44}$$

The stress is divided into a viscous tensor and pressure by subtracting the thermodynamic pressure.

$$\mathbf{T} = -p \delta + \tau \quad \text{A1.45}$$

The trace of the stress tensor forms a mechanical pressure force.

$$p_m = -\frac{1}{3} \text{tr}(\mathbf{T}) = -\frac{1}{3} T_{ii} \quad \text{A1.46}$$

Stokes assumption equates these pressures, $p = p_m$

The gravity force per unit volume \mathbf{g} where g is a constant scalar magnitude g times a unit vector in the gravity direction. If Z is the height above a horizontal reference plane,

$$\mathbf{F}_g = -\rho g \mathbf{Z} \quad \text{A1.47}$$

Two equivalent forms of the momentum are from the perspective of a fixed point in space or from the perspective of a material particle.

$$\begin{aligned} \frac{D(\rho \mathbf{v})}{Dt} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) &= -\nabla p + \nabla \cdot \tau + \rho \mathbf{g} \\ \rho \frac{D\mathbf{v}}{Dt} &= -\nabla p + \nabla \cdot \tau + \rho \mathbf{g} \end{aligned} \quad \text{A1.48}$$

An equation governing the mechanical energy or kinetic energy per unit mass is obtained by the dot product of velocity and the momentum equation.

$$\frac{D(\rho \frac{1}{2} v^2)}{Dt} + \nabla \cdot (\rho \mathbf{v} \frac{1}{2} v^2) = -\mathbf{v} \cdot \nabla p + \rho \mathbf{v} \cdot \mathbf{g} + \mathbf{v} \cdot (\nabla \cdot \tau) \quad \text{A1.49}$$

An equation governing the moment of momentum, $\mathbf{r} \times \mathbf{v}$, is obtained by \mathbf{r} cross the momentum equation.

$$\frac{D(\rho \varepsilon_{ijk} r_j v_k)}{Dt} + \rho \left[\varepsilon_{ijk} r_j \frac{Dv_k}{Dt} \right] = -\varepsilon_{ijk} r_j \frac{Dp}{Dt} + \rho \varepsilon_{ijk} r_j g_k + \varepsilon_{ijk} r_j \rho \tau_{ki}$$

A1.50

Any origin is permitted for \mathbf{r} .

The vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is governed by an equation formed by $\nabla \times$ momentum equation. The equation is

$$\frac{D\boldsymbol{\omega}}{Dt} = -\boldsymbol{\omega} \cdot \nabla \mathbf{v} + \boldsymbol{\omega} \cdot \nabla \mathbf{v} + \frac{1}{\rho^2} \nabla \rho \times \nabla p - \frac{1}{\rho^2} \nabla \rho \times \nabla \cdot \boldsymbol{\tau} + \frac{1}{\rho} \nabla \times \nabla \cdot \boldsymbol{\tau}$$

A1.51

(a derivation of this equation may be found in "Fluid Vortices," S. I. Green, Kluwer Academic Publishers, Dordrecht, Netherlands for complex fluids see "Fluid Dynamics of Viscoelastic Liquids," D. D. Joseph, Springer, 1990)

Angular momentum

Conservation of angular momentum is a distinct physical law from linear momentum.

The net internal angular momentum per unit mass is, $\tilde{\mathbf{a}}$. This would occur if the molecules were spinning in a preferred direction. Angular momentum crossing an imaginary surface by molecular transport (diffusion) would produce a surface couple $n_j j_i$. One could also propose an external physical process G_i that would impart angular momentum directly to the individual particles. Conservation of total angular momentum, $\mathbf{r} \times \mathbf{v} + \tilde{\mathbf{a}}$, leads to the equation

$$\frac{(\rho \epsilon_{ijk} r_j v_k + \rho \tilde{a}_i)}{t} + \rho v_p (\epsilon_{ijk} r_j v_k + \tilde{a}_i) \quad \text{A1.52}$$

$$= -\epsilon_{ijk} r_j \dot{p}_k + \rho \epsilon_{ijk} r_j g_k + \rho (\epsilon_{ijk} r_j \tau_{pk}) + \rho G_i + \dot{p}_j \tau_{jk}$$

Subtracting the moment of momentum equation derived earlier yields a relation governing internal angular momentum.

$$\frac{(\rho \tilde{a}_i)}{t} + \rho v_p \tilde{a}_i = \epsilon_{ijk} \tau_{jk} + \rho G_i + \dot{p}_j \tau_{jk} \quad \text{A1.53}$$

It is usually assumed that the molecular angular momentum is randomly distributed so that $\tilde{\mathbf{a}} = 0$ and \mathbf{G} and $\mathbf{\Omega}$ are zero. Then $\epsilon_{ijk} \tau_{jk} = 0$ and must be symmetric. Symmetry of τ will be assumed in the balance of section A.1.

Energy

The conservation law for total energy leads to

$$\frac{1}{t} \rho \left(e + \frac{1}{2} v^2 \right) + \dot{p} \cdot \rho \left(e + \frac{1}{2} v^2 \right) = - \dot{p} \cdot \mathbf{q} - \dot{p} \cdot (v p) + \rho v \cdot \mathbf{g} + \dot{p} \cdot (\boldsymbol{\tau} \cdot \mathbf{v}) \quad \text{A1.53}$$

Here the "heat" flux vector \mathbf{q} accounts for the transport of energy by microscopic mechanisms.

Subtracting the kinetic energy equation yields the thermal energy

$$\frac{1}{t} \rho e + \dot{p} \cdot (\rho v e) = - \dot{p} \cdot \mathbf{q} - p \dot{p} \cdot \mathbf{v} + \quad \text{A1.54}$$

The symbol $\boldsymbol{\tau} : \mathbf{v} = \tau_{ij} v_j$ represents the viscous dissipation that creates thermal energy from kinetic energy.

The thermal energy equation with temperature as the dependent variable is derived using general thermodynamic relations. It is

$$\rho c_p(T, p) \frac{DT}{Dt} = - \mathbf{q} \cdot \nabla T + \beta T \frac{Dp}{Dt} \quad \text{A1.55}$$

Entropy

The fundamental equation expressing the second law of thermodynamics is

$$\rho \frac{Ds}{Dt} = - \mathbf{q} \cdot \nabla \left(\frac{1}{T} \right) - \frac{1}{T^2} \mathbf{q} \cdot \nabla T + \frac{1}{T} \sigma \quad \text{A1.56}$$

The terms $-\mathbf{q} \cdot \nabla \left(\frac{1}{T} \right) - \frac{1}{T^2} \mathbf{q} \cdot \nabla T = -\frac{1}{T} \mathbf{q} \cdot \nabla T$ are written as two terms to

separate the reversible $-\mathbf{q} \cdot \nabla \left(\frac{1}{T} \right)$ and irreversible $-\frac{1}{T^2} \mathbf{q} \cdot \nabla T$ effects of heat

transfer. The viscous term is irreversible.

A1.4 Balance Laws and Global Governing Equations

Regions

Global laws are integrals of the local laws over a chosen region or, on the other hand, may be postulated as basic truths from the start. An arbitrary region, designated as AR, has a specified velocity \mathbf{w} at each point on the surface. For a material region, MR, the surface velocity is the local fluid velocity $\mathbf{w} = \mathbf{v}$. A fixed region, FR, is one with $\mathbf{w} = 0$ everywhere. It is sometimes useful to consider a region with a surface velocity constant in space $\mathbf{w} = \mathbf{W}(t)$. Such a region has a constant volume and is designated VR. The fluid velocity relative to the VR is

$$\mathbf{u} = \mathbf{v} - \mathbf{W} \quad \text{A1.57}$$

A region enclosing a rocket and following it through space is a volume region, VR. The velocity of the rocket is $\mathbf{W}(t)$ and the rocket engine discharges gases from the region.

Elementary Thermodynamic texts do not have a uniform notation for regions. The arbitrary region, AR, defined above might be called an "open system," a "deformable control volume," a "control volume," or some combination of these terms. The fixed region, FR, defined above might be called an "open system," or simply a "control volume." The material region might be called a "system," a constant mass system," or a "closed system."

Leibnitz and Gauss Theorems

For a region with arbitrary surface velocity \mathbf{w} , Leibnitz Theorem is

$$\frac{d}{dt} \int_{V_{AR}} f_{ij}(x_k, t) dV = \int_{AR} \frac{f_{ij}(x_k, t)}{t} dV + \int_{AR} n_m w_m f_{ij}(x_k, t) dS$$

A1.58

, and Gauss's theorem is

$$\int_{V_{AR}} \frac{\partial f_{jk}(x_l, t)}{\partial x_l} dV = \int_{AR} n_i f_{jk}(x_k, t) dS \quad \text{A1.59}$$

A global law is derived by letting f_{ij} in Leibnitz Theorem be the quantity of interest in the local equation, substituting the local equation for f_{ij} / t and converting as many volume integrals as possible into surface integrals by Gauss's theorem

Volume

The volume of the region changes with time according to

$$\frac{dV_{AR}}{dt} = \int_{AR} \mathbf{n} \cdot \mathbf{w} dS \quad \text{A1.60}$$

Here $\mathbf{n} \cdot \mathbf{w}$ is the normal velocity of the surface of the control region.

Mass

Conservation of mass for an arbitrary region is

$$\frac{d M_{AR}}{d t} = \frac{d}{d t} \int_{AR} \rho dV = - \int_{AR} \mathbf{n} \cdot (\mathbf{v} - \mathbf{w}) \rho dS \quad A1.61$$

Here all velocities are absolute velocities with respect to an inertial frame. For a material region, MR, $\mathbf{v} = \mathbf{w}$, and Eq. A1.61 becomes $\frac{d M_{MR}}{d t} = \frac{d}{d t} \int_{MR} \rho dV = 0$.

For a Volume Region, VR, with fluid velocity \mathbf{u} with respect to the moving region;

$$\frac{d M_{VR}}{d t} = - \int_{VR} \mathbf{n} \cdot \mathbf{u} \rho dS \quad A1.62$$

In the rocket example the mass changes because of the relative velocity of the gases leaving the rocket motor.

Linear Momentum

Arbitrary region

$$\begin{aligned} \frac{d}{dt} \int_{AR} (\rho \mathbf{v}) dV = & - \int_{AR} \rho \mathbf{n} \cdot (\mathbf{v} - \mathbf{w}) \mathbf{v} dS + \int_{AR_solid} \mathbf{n} \cdot \mathbf{T} dS \\ & + \int_{AR_fluid} \mathbf{n} \cdot \boldsymbol{\tau} dS - \int_{AR_fluid} \mathbf{n} p dS + \int_{AR_fluid} \rho \mathbf{F}_g dV \end{aligned}$$

A1.63

Moving volume region

$$\begin{aligned} \frac{d}{dt} \int_{VR} (\rho \mathbf{u}) dV + M_{VR} \frac{d \mathbf{W}}{dt} = & - \int_{VR} \rho (\mathbf{n} \cdot \mathbf{u}) \mathbf{u} dS + \int_{VR_solid} \mathbf{n} \cdot \mathbf{T} dS \\ & + \int_{VR_fluid} \mathbf{n} \cdot \boldsymbol{\tau} dS - \int_{VR_fluid} \mathbf{n} p dS + \int_{VR_fluid} \rho \mathbf{F}_g dV \end{aligned} \quad A1.64$$

Recall that the fluid velocity relative to the VR is $\mathbf{u} = \mathbf{v} - \mathbf{W}$.

Total energy

For the global energy equations the surface integrals are split into regions where the surface cuts a solid and where it cuts a fluid. Shaft work arises from solid surfaces that are cut by the control surface. Work of a rotating shaft, which involves the tangential velocity, v_t , or translating shaft, which involves the normal velocity, v_n , is described by this term.

$$\begin{aligned} \dot{W}_{Shaft} &= \int_{\text{Solid Surfaces}} (\mathbf{n} \cdot \mathbf{T}) \cdot \mathbf{v} dS = \int_{\text{Solid Surfaces}} (\mathbf{n} \cdot \mathbf{T}) \cdot (\mathbf{v}_n + \mathbf{v}_t) dS \\ &= \dot{W}_{Shaft, normal} + \dot{W}_{Shaft, rotary} \end{aligned} \quad A1.65$$

In fluid regions the stress tensor is decomposed into pressure and viscous parts,

$$T_{ij} = -p \delta_{ij} + \tau_{ij} \quad A1.66$$

The total energy equation is

$$\begin{aligned} \frac{d}{dt} \int_{AR} \rho \left(e + \frac{1}{2} v^2 + g Z \right) dV &= - \int_{AR} \rho \mathbf{n} \cdot (\mathbf{v} - \mathbf{w}) \left(e + \frac{1}{2} v^2 + g Z \right) dS + \dot{W}_{Shaft, normal} + \dot{W}_{Shaft, rotary} \\ &+ \int_{\text{Fluid Surfaces}} (\mathbf{n} \cdot \boldsymbol{\tau}) \cdot \mathbf{v} dS - \int_{\text{Fluid Surfaces}} p \mathbf{n} \cdot \mathbf{v} dS - \int_{AR} \mathbf{n} \cdot \mathbf{q} dS \end{aligned}$$

A1.67

This equation can be expressed in many forms. A popular form is to introduce the concepts of moving boundary work and flow work for the pressure work term.

$$\begin{aligned}
 - \int_{\text{Fluid Surfaces}} \rho \mathbf{n} \cdot \mathbf{v} \left(\frac{p}{\rho} \right) dS &= - \int_{\text{Fluid Surfaces}} \rho \mathbf{n} \cdot (\mathbf{v} - \mathbf{w}) \left(\frac{p}{\rho} \right) dS - \int_{\text{Fluid Surfaces}} \rho \mathbf{n} \cdot \mathbf{w} \left(\frac{p}{\rho} \right) dS \\
 &= \dot{W}_{FlowWork} + \dot{W}_{BoundaryWork}
 \end{aligned}$$

A1.68

Inserting these concepts into A1.67 allows the flow work to be incorporated into convective term and the enthalpy, $h = e + p/\rho$ to be identified.

$$\begin{aligned}
 \frac{d}{dt} \int_{AR} \rho \left(e + \frac{1}{2} v^2 + g Z \right) dV &= - \int_{AR} \rho \mathbf{n} \cdot (\mathbf{v} - \mathbf{w}) \left(h + \frac{1}{2} v^2 + g Z \right) dS + \dot{W}_{Shaft\ normal} + \dot{W}_{Sha\ rota} \\
 &\quad + \dot{W}_{Boundary} + \int_{\text{Fluid Surfaces}} (\mathbf{n} \cdot \boldsymbol{\tau}) \cdot \mathbf{v} dS - \int_{AR} \mathbf{n} \cdot \mathbf{q} dS
 \end{aligned}$$

A1.69

Thermal Energy

The global thermal energy equation may be found by integrating the local thermal energy equation.

$$\begin{aligned}
 \frac{d}{dt} \int_{AR} \rho e dV &= - \int_{AR} \rho n_i (v_i - w_i) e dS \\
 &\quad + \int_{AR} \tau_{ij} v_j dV - \int_{AR} n_i q_i dS + \int_{AR} p v_i dV
 \end{aligned}
 \tag{A1.70}$$

Mechanical Energy

Subtracting thermal energy from total energy yields the final general form for the compressible flow in an arbitrary region. Here the total head

$\left(\frac{1}{2} v^2 + g Z + \frac{p}{\rho} \right)$ appears. Note that some disciplines reserve the term "head"

for the items with the dimension of length; ie. $(\frac{1}{2g} v^2 + Z + \frac{p}{g\rho})$.

$$\frac{d}{dt} \int_{AR} \rho (\frac{1}{2} v^2 + g Z) dV = - \int_{AR} \rho \mathbf{n} \cdot (\mathbf{v} - \mathbf{w}) (\frac{1}{2} v^2 + g Z + \frac{p}{\rho}) dS + \dot{W}_{Shaft\ normal} + \dot{W}_{Shaft\ rotary} + \dot{W}_{Boundary} + \int_{Fluid\ Surfaces} (\mathbf{n} \cdot \boldsymbol{\tau}) \cdot \mathbf{v} dS + \int_{AR} dV - \int_{AR} p \cdot \mathbf{v} dV$$

A1.71

The term representing viscous dissipation is usually replaced by defining a headloss h_l .

$$\int_{FR} m g h_l = \int_{FR} \tau_{ij} v_j dV = \int_{FR} dV \text{ for symmetric } \tau$$

A1.72

Entropy

An exact expression of the second law of thermodynamics is

$$\frac{d}{dt} \int_{AR} \rho s dV = - \int_{AR} \rho n_i (v_i - w_i) s dS - \int_{AR} \frac{1}{T} n_i q_i dS - \int_{AR} \frac{1}{T^2} q_i v_i T dV + \int_{AR} \frac{1}{T} \tau_{ij} S_{ji} dV$$

A1.73

Here the last two terms are irreversible effects and always positive.

Neglecting them leads to an inequality well known in thermodynamics.

A1.5 Constitutive Equations

Many practically important fluids obey constitutive relations for a Newtonian fluid with Fourier heat conduction. The Newtonian stress-rate of strain relationship contains two viscosity coefficients

$$\tau_{ij} = \lambda \nabla_k v_k \delta_{ij} + 2 \mu S_{ij}$$

A1.74

Inserting Stokes assumption, $\tau_{ij} = -2\mu/3 \nabla_i v_j$

$$\tau_{ij} = -\frac{2}{3}\mu \nabla_k v_k \delta_{ij} + 2\mu S_{ij} \quad \text{A1.75}$$

The Fourier conduction law is

$$q_i = -\kappa \nabla_i T \quad \text{A1.76}$$

For completeness Fick diffusion law for a binary mixture relates the diffusion flux and the concentration gradient with the binary diffusion coefficient as the proportionality constant.

$$\mathbf{j}_A = \rho_A (\mathbf{v}_A - \mathbf{v}) = -\rho \mathcal{D}_{AB} \nabla x_A \quad \text{A1.77}$$

A1.6 Navier Stokes Equations

Navier-Stokes equations for a compressible flow may be considered as the continuity equation together with the momentum and energy equations for a Newtonian fluid.

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla \cdot \boldsymbol{\tau} = -\nabla p + \rho \mathbf{g} - \left(\frac{2}{3}\mu \nabla \cdot \mathbf{v}\right) \nabla + 2 \nabla \cdot (\mu \mathbf{S}) \quad \text{A1.78}$$

$$S = \frac{1}{2} \nabla \mathbf{v} + \frac{1}{2} (\nabla \mathbf{v})^T$$

$$\rho c_p(T, p) \frac{DT}{Dt} = -\kappa \nabla^2 T + \dot{q} + \beta T \frac{Dp}{Dt}$$

Incompressible

For incompressible flows the density is approximately constant and transport coefficients are approximately constant. These are consistent assumptions at low Mach numbers (a characteristic velocity divided by the speed of sound) with adiabatic walls or isothermal walls with small temperature differences .

The equations take the form

$$\boldsymbol{\omega} \cdot \mathbf{v} = 0 \quad \text{A1.79}$$

$$\frac{D\mathbf{v}}{Dt} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \boldsymbol{\omega} \times \mathbf{v} \quad \text{A1.80}$$

$$\boldsymbol{\omega} \times \mathbf{v} = \nabla^2 \mathbf{v}$$

$$\frac{D\boldsymbol{\omega}}{Dt} = -\boldsymbol{\omega} \cdot \nabla \mathbf{v} + \nabla^2 \mathbf{v} \quad \text{A1.81}$$

Two related equations govern the enstrophy and the pressure.

$$\frac{D}{Dt} \left(\frac{1}{2} \boldsymbol{\omega}^2 \right) = \omega_i \omega_j S_{ji} + v_j \omega_j \frac{1}{2} \boldsymbol{\omega}^2 - v_j \omega_i \omega_j \quad \text{A1.82}$$

$$\frac{1}{\rho} \nabla^2 p = \mathbf{v} \cdot \nabla^2 \mathbf{v} + \boldsymbol{\omega} \cdot \boldsymbol{\omega} - \nabla^2 \left(\frac{1}{2} v^2 \right) \quad \text{A1.83}$$

A1.7 Discontinuities in Density

Normal Surface Discontinuity

Consider a surface discontinuity in the fluid. This might be thought of as a region with a finite thickness in the limit as the thickness approaches zero. The surface moves with velocity \mathbf{W} . Figure A1-2 is a typical example that depicts a shock wave caused by a blunt body moving at supersonic speed with respect to the surrounding fluid. Fluid on one side is called fluid *A* and that on the other fluid *B*. The unit normal vector \mathbf{n}_A points from the discontinuity into fluid *A*, and \mathbf{n}_B points from the discontinuity into fluid *B*. Assume that there is no mass, momentum or energy within the discontinuity and that the tangential velocity component is unchanged, $\mathbf{v}_{At} = \mathbf{v}_{Bt}$. The mass flow across the discontinuity is conserved.

$$\rho_A \mathbf{n}_A \cdot (\mathbf{v}_A - \mathbf{W}) + \rho_B \mathbf{n}_B \cdot (\mathbf{v}_B - \mathbf{W}) = 0 \quad \text{A1.84}$$

The normal vectors may be replaced, $\mathbf{n}_A = -\mathbf{n}_B = \mathbf{n}$. The tangential momentum equation yields a balance of shear forces.

$$(\mathbf{n}_A \cdot \boldsymbol{\tau}_A)_t + (\mathbf{n}_B \cdot \boldsymbol{\tau}_B)_t = 0 \quad \text{A1.85}$$

Let the normal component of the fluid velocity be $\mathbf{n}_A \cdot \mathbf{v}_A = v_{An}$. The normal direction momentum equation is

$$\begin{aligned} & [\rho_A \mathbf{n}_A \cdot (\mathbf{v}_A - \mathbf{W}) v_{An} + p_A - \mathbf{n}_A \cdot (\mathbf{n}_A \cdot \boldsymbol{\tau}_A)] \\ & + \mathbf{n}_B \cdot [\rho_B \mathbf{n}_B \cdot (\mathbf{v}_B - \mathbf{W}) v_{Bn} + p_B - \mathbf{n}_B \cdot (\mathbf{n}_B \cdot \boldsymbol{\tau}_B)] = 0 \end{aligned} \quad \text{A1.86}$$

If e_t is the total (kinetic plus internal) energy, The conservation of energy requires that

$$\begin{aligned} & [\rho_A e_{tA} \mathbf{n}_A \cdot (\mathbf{v}_A - \mathbf{W}) v_{An} + p_A v_{An} - \mathbf{n}_A \cdot (\boldsymbol{\tau}_A \cdot \mathbf{v}_A) + \mathbf{n}_A \cdot \mathbf{q}_A] \\ & + [\rho_B e_{tB} \mathbf{n}_B \cdot (\mathbf{v}_B - \mathbf{W}) v_{Bn} + p_B v_{Bn} - \mathbf{n}_B \cdot (\boldsymbol{\tau}_B \cdot \mathbf{v}_B) + \mathbf{n}_B \cdot \mathbf{q}_B] = 0 \end{aligned} \quad \text{A1.87}$$

Because the discontinuity surface contains no mass, the curvature or the time dependence of \mathbf{W} have no effect on the local validity of the equations above.

Fluid-Solid Boundary

The central issue here is the condition on the tangential velocity. In many situations the viscous no-slip condition is adequate. However, some situations require a more refined approach. The analysis above covers the case of a porous or vaporizing wall without slip. Let u_s be the slip velocity of the fluid along the wall. Slip is often expressed as a slip length defined as

$$\beta = \frac{u_s}{\dot{\gamma} \left. \frac{du}{dy} \right|_0} = \frac{u_s}{\dot{\gamma}} \quad \text{A1.88}$$

$$\dot{\gamma} \left. \frac{du}{dy} \right|_0 \quad \text{A1.89}$$

Because the interactions of gasses with solids and liquids with solids are different, these fluids must be dealt with separately.

In gases an important parameter is the Knudsen number that compares the mean-free-path length λ to the flow length scale h .

$$Kn = \frac{\lambda}{h} = \sqrt{\frac{\gamma \pi}{2}} \frac{M}{Re} \quad A1.90$$

The relation with the Mach and Reynolds numbers is often useful. A first order equation (derived from kinetic theory) for the slip is Maxwell's equation.

$$u_s = \frac{2 - \sigma_v}{\sigma_v} Kn h \dot{\gamma} \quad A1.91$$

Here σ_v is the tangential momentum accommodation coefficient. The Knudsen number becomes large for large object in a rarefied gas, or a very small object at atmospheric pressure.

Slip in liquids is only observed at small scales. The slip length is on the order on 100 nanometers (0.1 micrometers) Experimental results have been correlated as a power law

$$\beta = A \dot{\gamma}^B \quad A1.92$$

$$u_s = A \dot{\gamma}^{B+1}$$

The constant B is about 1/2. Unfortunately, the dimensions in the constants in the expressions above interact. Changing B changes the dimensions of A . More detailed comments are in chapter C15.

Interfaces with Surface Tension

Interfaces with two thermodynamic phases or immiscible substances may require that we postulate a surface tension property. Surface tension, force per length, is the two-dimensional world analogue to pressure. However, it is taken as a thermodynamic property of the substances and the temperature. The curvature of the surface is important. Let R_1 and R_2 be the principle radii of curvature of the surface. The curvature is

$$2 \mathcal{H} = - \frac{1}{R_1} + \frac{1}{R_2} \quad \text{A1.93}$$

Conservation of mass leads to the same equation as for normal discontinuities. The momentum equation contains two surface tension effects. One effect is from the curvature of the surface and another from a possible variation of the surface tension along the surface.

$$\begin{aligned} \rho \mathbf{n} \cdot (\mathbf{v}_A - \mathbf{w}) \mathbf{v}_A - \rho \mathbf{n} \cdot (\mathbf{v}_B - \mathbf{w}) \mathbf{v}_B \\ = [\mathbf{n} \cdot \boldsymbol{\tau}_A - \mathbf{n} p_A] - [\mathbf{n} \cdot \boldsymbol{\tau}_B - \mathbf{n} p_B] - \sigma \mathcal{H} \end{aligned} \quad \text{A1.94}$$

Surfaces without mass crossing the interface have simplified expressions. In addition to the unit vector \mathbf{n} normal to the surface, let the vectors \mathbf{b} and \mathbf{t} be orthogonal unit vectors within the surface. The momentum equation in these three directions is

$$\mathbf{n}\text{-direction:} \quad 0 = [\tau_{n n_A} - \tau_{n n_B}] - [p_A - p_B] - 2\sigma \mathcal{H} \quad \text{A1.95}$$

$$\mathbf{t}\text{-direction:} \quad 0 = [\tau_{t t_A} - \tau_{t t_B}] - \frac{d\sigma}{dt} \quad \text{A1.96}$$

$$\mathbf{b}\text{-direction} \quad 0 = [\tau_{b b_A} - \tau_{b b_B}] - \frac{d\sigma}{db} \quad \text{A1.97}$$

Contact lines, where two surfaces meet, require special treatment and if the contact line moves slip is required.

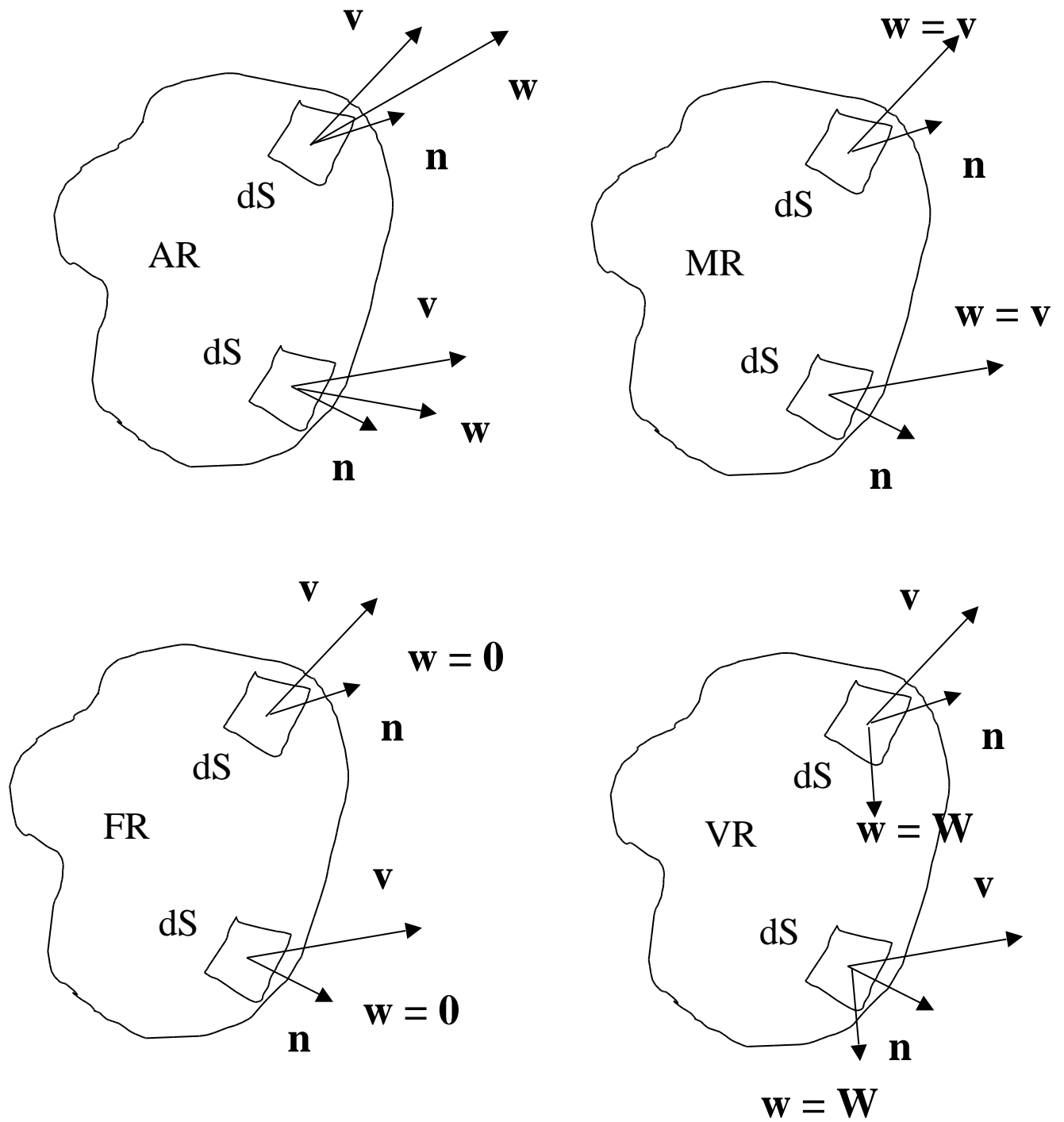


Figure A1-1: AR, Arbitrary region with \mathbf{w} prescribed; MR, Material region with \mathbf{w} equal the local fluid velocity \mathbf{v} ; FR, Fixed region with \mathbf{w} equal zero; VR, Volume region with \mathbf{w} equal a constant in space $\mathbf{W}(t)$

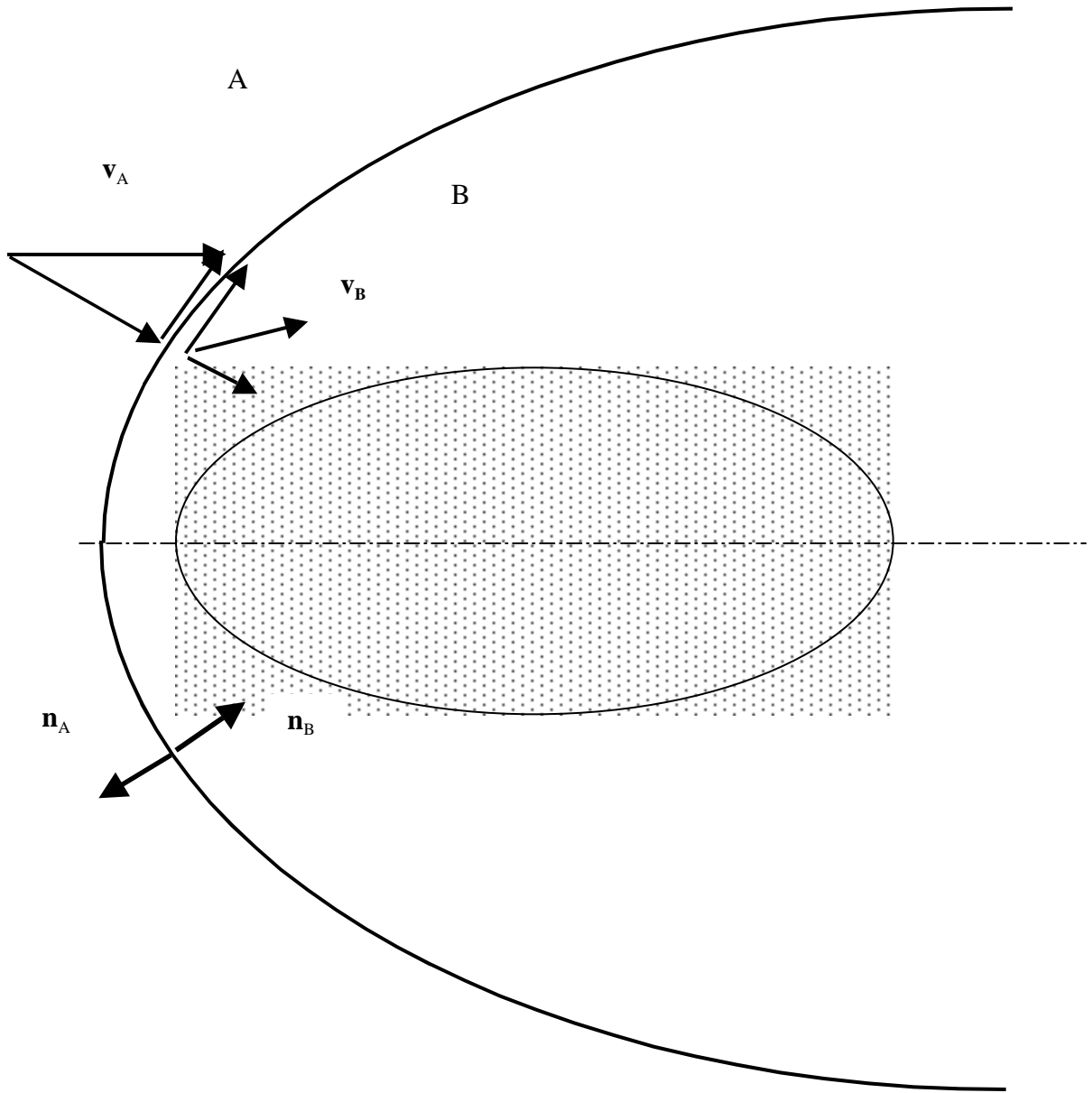


Figure A1-2 Discontinuity across shock wave. Tangential velocity is unchanged. Normal velocity is decreased.