SOME ISSUES CONCERNING WALL TURBULENCE

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Introduction
Recently, several different ideas about turbulent wall layers have been put forward. In each case the orthodox viewpoint is challenged and/or modified. Five topics will be addressed:
1. Barenblatt’s power law vs log law for (a) pipe flow and (b) boundary layer flows
2. Varying log law constants and shift of \( y \) origin
3. Three layer structure
4. Defect law scale \( U_o - U_{ave} \)
5. von Kármán constant and beginning of log law
6. Scaling of \( u^2(y) / u_*^2 \)

Before these ideas are discussed a modern version of the classical two-layer theory will be given. This background will give a better point from which to view the issues.

Background
Theories on turbulence are necessarily incomplete. Tennekes and Lumley (1972) have composed a modern version of turbulent wall layer theory based on asymptotic behavior at large Reynolds numbers. An alternative version is in Panton (1996).

Consider the fully developed flow in a slot formed by plane walls. The essential features of this flow are the same as those for round pipes. The transverse coordinate \( y \) is measured from the lower wall and the half-height \( h \) is the centerline where the velocity is \( U_o \). The theory developed for the asymptotic case \( Re_* = u* h / \nu \Rightarrow \infty \). The friction velocity is \( u_* \).

The outer region variable is
\[
Y = \frac{y}{h}
\]

It is assumed that the velocity and stress profiles can be represented by Poincaré expansions that separate the \( Y \) and \( Re_* \) dependence into multiplicative functions.
\[
\frac{U(Y)}{U_o} = F(Y, Re_*) \sim F_0(Y) + \Delta_1(Re_*) F_1(Y) + \Delta_2(Re_*) F_2(Y) + ... \quad (2)
\]
and
\[
- \frac{uv}{u_*^2} = G(Y, Re_*) \sim G_0(Y) + ... \quad \text{as } Re_* \Rightarrow \infty \quad (3)
\]

Here \( \Delta_i(Re_*) \) is a sequence of gauge functions (\( \Delta_1 \Rightarrow 0 \) as \( Re_* \Rightarrow \infty \)).

The outer momentum equation shows that the Reynolds stress function is
\[ G_0 = 1 - Y \quad (4) \]
Arguments employing the mean kinetic energy equation to estimate dominant terms conclude that \( \Delta_1 = u_* / U_o \) and that

\[ G_0 \frac{dF_0}{dY} = 0 \quad \text{thus} \quad F_0 = 1 \quad (5) \]

Rearranging Eq.(2) shows that \( F_1 \) is the \textit{defect law}

\[ F_1(Y) = \frac{U(y) - U_o}{u_*} \quad (6) \]

Next, turn attention to the inner layer. The inner distance variable is

\[ y^+ = \frac{u_* y}{v} = Y \text{Re}_* \quad (7) \]

Inner layer Poincaré expansions are

\[ \frac{U}{u_*} = f(y^+, \text{Re}_*) - f_o(y^+) + ... \quad \text{as} \quad \text{Re}_* \to \infty \quad (8) \]

and

\[ -\frac{uv}{u_*^2} = g(y^+, \text{Re}_*) - g_o(y^+) + ... \quad \text{as} \quad \text{Re}_* \to \infty \quad (9) \]

Rigorous matching between inner and outer expansions is done in an intermediate variable. This is called \textit{limit process matching}. Let

\[ \eta \equiv \text{Y Re}_*^\alpha \quad 0 \leq \alpha \leq 1 \quad (10) \]

If \( \alpha = 0, \eta = Y \); if \( \alpha = 1, \eta = y^+ \). For \( 0 < \alpha < 1, \eta \) is an intermediate variable. In the limit \( \text{Re}_* \to \infty \) with \( \eta \) fixed \( (0 < \alpha < 1) \) the inner and outer expansions are asymptotically equal.

\[ \{ 1 + \frac{u_*}{U_o} F_1(Y \Rightarrow \eta \text{Re}_*^{-\alpha}) + ... - \frac{u_*}{U_o} \left[ f_o(y^+ \Rightarrow \eta \text{Re}_*^{-1-\alpha}) + ... \right] \} = 0 \quad (11) \]

Since \( \text{Re}_* \to \infty \) with \( \eta \) fixed means that \( Y \to 0 \) and that \( y^+ \to \infty \), it is common to write; \( F(Y \Rightarrow 0) = \frac{u_*}{U_o} f(y^+ \Rightarrow \infty) \).

The parts of the functions that match are called \textit{common parts} and denoted by subscript \( cp \). Differentiating the matching equation yields

\[ Y \frac{dF_1 \text{ cp}}{dY} = y^+ \frac{df_o \text{ cp}}{dy^+} = \frac{1}{\kappa} \quad (12) \]

Regarding \( Y \) and \( y^+ \) as independent and noting that \( F_1 \) and \( f_o \) cannot depend upon \( \text{Re}_* \) means that \( \kappa = \) constant. The solutions are given below together with the common parts for the Reynolds stress functions.

Inner region

\[ f_o \text{ cp}(y^+) = \frac{U(y)}{u_*} = \frac{1}{\kappa} \ln y^+ + C_i \quad (13) \]
\[ g_o \text{ cp}(y^+) = 1 \quad (14) \]
Subtracting the velocity expressions gives

\[
\frac{U_0}{u_*} = \frac{1}{\kappa} \ln (Re_*) + C_i - C_o
\]  

(17)

The scaling for \( u_* \) as a function of the Reynolds number is a result of matching. The numerical value of \( \kappa \) is fixed by setting \( u_* \) equal to the friction velocity \( u_c \). An alternative would have been to set \( \kappa = 1 \) and the wall stress would have been experimentally determined.

It is possible to form composite expansions that are uniformly valid for all \( y \). An additive composite is formed by adding the inner and outer expansions and subtracting the common part. For the mean velocity we have

\[
\frac{U(y)}{u_*} = f_o(y^+) + F_1(Y) - [F_1(Y)]_{cp}
\]  

(18)

Next the law of the wake is defined as

\[
W(Y) = F_1(Y) - [F_1(Y)]_{cp} = F_1(Y) - \left[ \frac{1}{\kappa} \ln Y + C_o \right]
\]

This gives a more compact form

\[
\frac{U(y)}{u_*} = f_o(y^+) + W(Y) \quad \text{where} \quad Y = y^+ / Re_*
\]  

(19)

Together the law-of-the-wall and the law-of-the-wake form a uniformly valid representation for the mean velocity and show the first order variation with \( Re_* \).

Using equivalent logic (Panton(1996), a composite expansion for the Reynolds stress is

\[
-\frac{uv(y)}{u_*^2} = g_o(y^+) + G_o(Y) - G_{o-cp}(Y)
\]  

(20)

Since \( G_o = 1 - Y \), the common part is \( G_o(Y\Rightarrow 0) = 1 \). Hence the “wake” function for the Reynolds stress is

\[
G_o^{(w)}(Y) = G_o(Y) - G_{o-cp}(Y) = -Y = y^+ / Re_*
\]

Thus, the final uniformly valid expansion is

\[
-\frac{uv(y)}{u_*^2} = g_o(y^+) - \frac{y^+}{Re_*}
\]  

(21)

Experimental data from pipe and channel flows and DNS results correlate very well when Eq. 21 is used to isolate \( g_o \). An empirical relation used for \( g_o \) has been fitted to the data. Figure 1 is a plot of the Poincaré composite expansion for \( Re_* \) values from 200 to 100,000. The two empirical constants are taken as \( \kappa = 0.41 \) and \( C_i = 5.25 \).
As an aside, note that any active motion component (by definition a motion that contributes to the Reynolds stress) should have similarity in the inner region. Thus, they may be represented by a composite expansion of the form

\[ X(y^+, \text{Re}_*) = x_o(y^+) + X^{(w)}(Y=y^+/	ext{Re}_*) \]  

(22)

Here the “wake” component is the outer function minus the common part:

\[ X^{(w)}(Y) \equiv X_o(Y) - X_{o,cp}(Y) \]

This sum of an inner law and a “wake law” is uniformly valid. It shows the first order variation with Reynolds number.

Based on the behavior of maximum \( \overline{uv} \) at low Reynolds numbers Wei and Willmarth (1992) speculated that inner layer scaling was lost and the physical processes modified. Figure 1 shows that when the inner layer becomes a large fraction of the outer, a simple mixing of inner and outer layer by a Poincaré composite expansion adequately explains the observed trends to \( \text{Re}_* \).

The Poincaré two-region theory does not actually require three functions; \( f_o(y^+) \), \( W(Y) \) and \( g_o(y^+) \). The momentum equation requires that

\[ f_o = \int_0^{y^+} (1 - g_o) \, dy^+ \]  

(22)

Figure 2 plots Eq. (19) for the mean velocity for the same range of \( \text{Re}_* \) values. The \( f_o \) relation is the integral of \( g_o \) used in Fig (2); constants are \( \kappa = 0.41 \) and \( C_1 = 5.25 \) as before. The wake-law is Coles’ sine function with the corner correction of Lewkowicz(1982). The strength of the wake contribution is taken as \( C_o = 1.0 \). This is the value recommended by Tennekes and Lumley (1972) for round pipes.

Several things are worthy of note about Figs. 1 and 2. The common part of the velocity profile, the log function, begins to separate out at about \( \text{Re}_* = 500 \). The Reynolds number needed to closely approach one decade of the log law is \( \text{Re}_* = 3000 \) (\( \text{Re}_D = 1.3 \times 10^5 \)). On the other hand, the Reynolds stress common part, \( g_o = 1 \), is not as closely approached. At \( \text{Re}_* = 300 \), the maximum \( -\overline{uv}/u_*^2 = 0.85 \) and at \( \text{Re}_* = 3000 \) it is only 0.94. A region where \( -\overline{uv}/u_*^2 \sim 1 \) does not appear until a much higher; \( \text{Re}_* = 10000 \). This just happens to be the nature of the functions involved. The maximum value is much less than one and moves outward (in \( y^+ \) units) as \( \text{Re} \) increases. It results from a simple mixing of a monotonically increasing inner function \( g_o \) and a monotonically decreasing outer function \( G_o \). The theoretical position of the maximum is \( \sim \text{Re}_*^{1/2} \) as \( \text{Re}_* \) increases.

In Fig. 3 the Poincaré composite velocity relation is compared to experimental data of Zagarola and Smits(1997). The constants were maintained to be the same as in Fig 2. A slight modification of the constants would produce a better fit, however, no attempt was made to optimize. The important point is that the shape of the composite expansion and the data are very much alike. The data drop off at large values of \( y^+ \) because data was taken across the centerline; the center of the pipe is \( y^+ = 102,190 \). In Fig. 4 data from DNS is compared to the Poincaré composite velocity. These data are at the other extreme of Reynolds numbers; \( \text{Re}_* = 180 \). Again no attempt was made to adjust the constants. Even though no log region exists the composite expansion has roughly the proper shape. At this low \( \text{Re}_* \) second order effects are possible. Figures 3 and 4 indicate that a composite velocity profile can represent the data over the entire range of \( y^+ \) and \( \text{Re}_* \).
Some additional comments are in order. Historically, in the first derivations of the log law, some physical assumptions about the turbulence were made, the term “constant stress” was emphasized by Townsend, for example. The assumption "production equals dissipation." is another common statement. One is tempted to think that because the answer is true that all the assumptions must be true. Or, on the other hand, if the assumption is not true, the answer is not true. This is not so; the arguments are so approximate that there are compensating errors. The Reynolds stress is not really constant anywhere, and production = dissipation only roughly. These terms should not be used to characterize the log region as they are only asymptotically true. The Millikan-Izakson argument produced the log law without any assumptions about the structure of turbulence.

1. Barrenblatt's power law vs log law

Barenblatt proposes that a power law is superior the log law. Specifically he advocates the relation

\[ \frac{U(y)}{u_*} = Cy^n \]  
(23)

where the coefficients are functions of the pipe Reynolds number, \( Re_d \)

\[ n(Re_d) = \frac{a}{\ln Re_d} \quad ; \quad a = \frac{3}{2} \]  
(24)

\[ C(Re_d) = c + d \ln Re_d \quad ; \quad c = \sqrt{\frac{1}{3}}, \quad d = \frac{5}{2} \]  
(25)

The choice of the coefficients \( a, b, \) and \( c \) was based on Nikuradse's data. These equations should not be confused with the power law used in engineering approximations;

\[ \frac{U}{U_o} = Y^n \]  
(26)

Barenblatt does not intend that (23) is valid in the center of the pipe or near the wall.

In order not to misrepresent Barenblatt's opinion I will quote directly from three papers.


"We re-emphasize that neither the power law nor the logarithmic law should be considered merely as convenient representations of empirical data. Rather they have equally rigorous theoretical foundations which are, however, based on essentially different assumptions.

An important question of a qualitative nature arises, therefore, as to whether either of these assumptions is correct?

The results presented in this work give some evidence in favor of the scaling law (power law)....."


"Therefore neither the logarithmic law nor the power law should be considered only as convenient representations of experimental data; both have rigorous foundations based, however, on different assumptions......

Therefore an important qualitative question arises: which of these assumptions is correct?

The results presented below (Barenblatt, 1991, 1993a; Barenblatt and Prostokishin, 1993) give some evidence in favor of the power law....."


"In a previous paper,.... it was argued that the von Kármán-Prandtl law is not appropriate and that a correct description is given by the scaling (power) law;....

The goal of this present paper is to examine what the experimental data, in particular recent experimental data, tell us about the validity of one or the other of the contrasting laws..."
The analysis of new experimental data adduces arguments against the von Kármán-Prandtl universal logarithmic law and in favor of a specific power law.

There are three statements or implications in Barenblatt's writings that need to be examined.

First: The log and power laws should be compared to each other.
Second: Only one law is "correct."
Third: The power law has a rigorous theoretical foundation.

Barenblatt's equations have been plotted for \( \text{Re}^* = 200, 1000, 10,000, \) and \( 100,000 \) along with the Poincaré composite expansion in Fig. 5 (see also Figs. 3, 4 and 6). The power law starts at a level higher than the log law, comes down to be roughly tangent to the log law and then rises. When the center of the pipe is reached the power law has its maximum slope. The power law is not intended to be valid near the wall or in the center of the pipe. As the Reynolds number increases the region where the power law is tangent to the log law moves outward. Thus, the region of invalidity near the wall increases in size. The major point of Fig. 5 is that the power law mimics the outer part of the log law and the beginning of the wake law. The power law gives up a region near the wall but matches further out than does the log law. It includes the first portion of the wake region. Thus, it is improper to compare the power law solely to the log law. In the Poincaré theory the log law is merely the limiting behavior \( \frac{u_*}{U_0} = f(y^+ \to \infty) \) or \( F(Y \to 0) \). Another point is that there is a region in \( y^+ \) where the power law does a reasonable job of representing the data. However, the power law is not uniformly valid as is the Poincaré composite expansion.

Authors of recent pipe flow measurements, Zagarola and Smits(1997) and Toonder and Nieuwstadt(1996), have been asked to compare their data to the power law. Neither set of authors concludes that the power law is a superior representation. More recently Osterlund et al (2000) concluded that their boundary layer measurements did not fit a power law for \( y^+ < 900 \).

Barenblatt begins his derivation from the equation
\[
\frac{d}{dy^+} y^+ \frac{U}{u^*} = \phi(y^+, \text{Re})
\]  
(27)

This statement is equivalent to the relation \( \frac{U}{u^*} = \phi_1(y^+, \text{Re}) \) that comes from a dimensional analysis of the problem. The Prandtl-von Kármán heuristic derivation of the log law included assumptions of an eddy viscosity, a constant Reynolds stress equal to the wall shear, and a mixing length increasing linearly from the wall. With these assumptions the right side of Eq. (27) is a constant; the same relation as Eq. (12). Alternately the Millikan-Izackson derivation produces Eq. (12) directly as the equation governing the common parts of matched Poincaré expansions. This renders the heuristic derivation moot. In any event Barenblatt calls it complete similarity if \( \phi(y^+, \text{Re}^*) = 1/\kappa \). To go further he sets the right side of Eq. (27) equal to a power law (with coefficients that depend on Reynolds number), and call this incomplete similarity
\[
\frac{d}{dy^+} y^+ \frac{U}{u^*} = C \kappa \gamma^{1+n}
\]  
(28)

This is a assumption about the mathematical form of \( \phi(y^+, \text{Re}) \). The only rigorous theoretical assumption seems to be the dimensional analysis used to find \( \frac{U}{u^*} = \phi_1(y^+, \text{Re}) \). Integration of (33) produces (23). The functional forms of the coefficients \( C(\text{Re}) \) and \( n(\text{Re}) \) in (24) and (25) are arbitrary.

In contrast, the two-layer Poincaré expansion theory uses the physical differential equations, deals simultaneously with Reynolds stresses and velocity, produces \( \phi(y^+, \text{Re}^*) = 1/\kappa \) because of the matching process, and yields uniformly valid results that show Reynolds number dependence.
The Barenblatt power law is not a Poincaré expansion. Some problems are so mathematically complicated that Poincaré expansions are not useful and more general expansions are needed. Many examples are given in Barenblatt (1996). In other instances, either type is useful. Recall that the wake components in channel flow and boundary layer flow are different from that in cylindrical pipe flow. Thus, the Barenblatt pipe power law, because it includes part of the wake region, does not directly apply. In Barenblatt et al. (1997b) and Barenblatt et al. (2000) attempts to fit the power law to a zero pressure gradient boundary layer are given. The first paper analyzes the data of Nagib and Hites (1995), while the second analyzes the data of Osterlund and Johansson (1999). The approach is to use the same functions and coefficient in Eqs 24 and 25, with a redefined Reynolds number. Thus, they are defining a new length scale $\Lambda$ for boundary layer that will give a Reynolds number, $U\lambda/\nu$, to make Eqs. (24) and (25) correct. Their method is to determine $A$ and $n$ from the data. Then they solve Eq. (24) for one estimate of the effective Reynolds number, $Re_1$, and to solve Eq. (25) for another estimate, $Re_2$. If the two Reynolds numbers are the same, then a unique length has been found. Figure 7 shows the per cent difference in the logarithms of the two estimates (as given by Barenblatt). The IIT data shows a consistent difference that averages 2.7%. The Stockholm data shows a high value at low $Re_\theta$, decreasing almost linearly with Reynolds number. Barenblatt et al. claim success for $Re_\theta > 10,000$. They quote a 3% difference in the $(ln Re_2 - ln Re_1)/ln Re$ (here $Re$ is $\sqrt{Re_1*Re_2}$) as sufficient and reasonable agreement. However, if one converts 3% increase in $ln Re_1$ into the Reynolds number itself, some large errors result. For example if $Re=5000$ the difference is 29.1%, for $Re=30,000$ the difference is 36.2%. This would be the error in the definition of $\Lambda$. Success in $ln X$ is not success in $X$. A better result would require changing the constants or changing the $Re$ functional forms in Eqs (24) and (25). They are not fixed by any theory.

In Barenblatt et al. (1997b) they find the ratio of $\Lambda/\Theta$ is approximately constant at 3. However, in Barenblatt et al. (2000) they state “Naturally the ratio of two length scales $\Theta/\Lambda$ is different for different runs: both these quantities depend upon the details of the flows, in particular they can depend in principle upon the distance between the tip of the plate and the point of observation.” For a properly defined length, the ratio $\Lambda/\Theta$ should be at most a function of $Re_*$. The fact that pipe and boundary layers need different constants is actually further evidence that the power law and the log law should not be compared. In principle the log law is the asymptotic behavior of all inner layers and should apply to channels, tubes, and boundary layers. Note that Barenblatt and coworkers need to revise the constants again for channel flows.

Summary

• The Barenblatt power law models the outer part of the log region and the inner part of the wake region. Since the power and log laws are valid in different regions it is not appropriate to ask which law is correct.
• The only theoretical bases used in deriving the power law is dimensional analysis which shows that $U/u'_* = \phi_1(y^+, Re_*)$. The functions $C(Re_*)$ and $n(Re_*)$ in (24) are arbitrary.
• The extension of the power law to boundary layers by defining a new length scale was unsuccessful.
• The Barenblatt laws are sophisticated three-constant curve fitting to part of the outer region (log+wake) of turbulent pipe flow.

2. Variable Constants (higher math) and Origin Shift

George et al. (1997) begin by assuming the two-layer structure where the inner is
\[ \frac{U}{u_*} = f(y^+, \text{Re}_*) \]  

and the outer has the defect form

\[ \frac{U(y) - U_o}{u_*} = F_1(Y, \text{Re}_*) \]  

Their analysis does not use Poincaré expansions. It proceeds with an asymptotic analysis that has two unique features. The first major assumption, called the "Asymptotic Invariance Principle," is that the form of these functions have finite values in the limit as \( \text{Re}_* \to \infty \) (This is not true for the Barenblatt expression), and are equal to the usual Poincaré type expansions. As \( \text{Re}_* \to \infty \) (29) and (30) become

\[ \frac{U}{u_*} = f(y^+, \text{Re}_*) \sim f_\infty(y^+) \text{ as } \text{Re}_* \to \infty \]  

\[ \frac{U(y) - U_o}{u_*} = F_1(Y, \text{Re}_*) \sim F_1\infty(Y) \text{ as } \text{Re}_* \to \infty \]  

\( f_\infty(y^+) \) and \( F_1\infty(Y) \) are the usual log laws. The second major assumption is that \( f(y^+, \text{Re}_*) \) and \( F_1(Y, \text{Re}_*) \) can be matched when \( y^+ \to \infty, Y \to 0 \) and Re a finite constant. Thus, the log law coefficients are functions of Reynolds number. They call this matching procedure "Near-Asymptotics" because it is not the usual intermediate variable matching.

With some assumptions about the \( \text{Re}_* \) behavior, the resulting relations are

\[ y^+ \frac{df}{dy^+} = \frac{1}{\kappa} \quad \text{and} \quad Y \frac{dF_1}{dY} = \frac{1}{\kappa} \]  

George et al. note that these equations are invariant to a transformation \( y^+ \to y^+ + a^+ \) and give solutions

\[ f = (1/\kappa(\text{Re}_*)) \ln (y^+ + a^+(\text{Re}_*)) + C_1(\text{Re}_*) \]  

and

\[ F_1 = (1/\kappa(\text{Re}_*)) \ln (Y + A(\text{Re}_*)) + C_0(\text{Re}_*) \]  

They conclude that the origin shift parameter \( a^+ \) is associated with a mesolayer that exists from 30 < \( y^+ < 300 \). The reason a mesolayer is indicated by the presence of \( a^+ \) is unclear. Note that George’s mesolayer is part of the inner layer and not a third layer.

There is, however, a serious comment concerning the existence of the constants \( a^+(\text{Re}_*) \) and \( A^+(\text{Re}_*) \). The fact that Eq. (33) is invariant to the transformation \( \xi = y^+ + a^+ \) means that \( f(\xi) \) obeys an equation of the same form; i.e. \( \xi \frac{df(\xi)}{d\xi} = 1/\kappa \). It does not mean that \( f(\xi) \) obeys the matching equation \( y^+ \frac{df}{dy^+} = 1/\kappa \). Indeed, Eqs. (34) and (35) are not solutions of (33) unless \( a^+(\text{Re}_*) \) and \( A^+(\text{Re}_*) \) equal zero. The addition of \( a^+(\text{Re}_*) \) and \( A^+(\text{Re}_*) \) is essentially an arbitrary generalization of the log laws.
A side issue is their statement (p14) that quantities (such as Reynolds stress) that have only a single velocity scale (in inner and outer regions) have a logarithmic matching relation. In "Incompressible Flow," 2ed, Section 15.7, one finds that matching to a constant (the Prandtl type matching) comes when the dependent variable has the same normalization scale in inner and outer regions. A log overlap form comes about from matching a defect form containing a constant reference $U_0$. In fact, Millikan remarks in his paper that the power law comes if the outer form $U/U_0$ (no defect) is matched to $U/u^*$.

In the end, theory aside, one can say: A first order function was found using a Poincaré expansion. For an expanded range of accuracy, let the coefficients be dependent on Re. In approximation a function of several variables there is a lot of flexibility. The Poincaré type expansion is what van Dyke would call a rational scheme because it can in principle be continued to higher orders. When one takes a Poincaré type expansion and subsequently allows the coefficients to be functions of the parameter, one gives up some nice mathematical properties, but with the greater flexibility should fit data more accurately. Lagerstrom (p124) gives an example where one cannot match two Poincaré expansions without allowing a coefficient to be a function of the parameter. Thus, it is something of a last resort to abandon the Poincaré type expansion, but there are precedents. Another paper that tends to validate the idea of allowing the coefficients to be functions of $Re^*$ is by Afzal(1976). Afzal produced a second-order Poincaré expansion. Collecting first order and part of the second order terms together makes the first order coefficients functions of Reynolds number.

Summary
• Allowing log law coefficients to be functions of $Re^*$ removes the functions from the category of a rational Poincaré sequence. One might view this as ad hoc means of accounting for part of the next higher order approximation. Variable constants should allow a better fit.
• The $y$ origin shifting constant $a^+(Re^*)$ is not theoretically founded. The overlap laws containing $a^+$ (and $A^+$) are not solutions of the overlap relations obtained by matching. The addition of $a^+(Re^*)$ is an arbitrary generalization of the log laws.

3. Three Layer Structure
There have been several proposals for a three layer structure. That is, regions where the independent variable has three distinct scalings; say $y/h$, $y/(h\sqrt{Re^*})$, and $y/(hRe^*)$. Recently, Sreenivasan and Sahay (1997) have attached special physical significance to the behavior of the maximum $\bar{uv}$ and speculated that another region exist in this neighborhood $\eta^{1/2} \equiv Y Re^{1/2}$. In singular perturbation problems the need for an intermediate layer would be indicated by the inability to match the two layers and/or the existence of a distinguished limit (a limit that produces a distinct balance of terms) in the governing equations. Apparently the matching of both Reynolds stress and mean velocity is satisfactory. Let us consider the equations governing the problem; the integrated momentum equation and the unknown Reynolds stress relation.

\[
\begin{align*}
- \bar{uv} (y) &+ \nu \frac{dU}{dy} = u^* \alpha[1 - \frac{y}{h}] \quad (36) \\
- \bar{uv} (y) &= \text{unknown governing relation} \quad (37)
\end{align*}
\]

As is well known, there is a differential equation for $\bar{uv}$, however, it just shifts the indeterminacy to other variables. In non dimensional form the momentum equation is

\[
\begin{align*}
- \bar{uv} (y) &+ Re^* \alpha-1 \frac{dU/u^*}{d\eta} = 1 - Re^*^{-\alpha} \eta \quad (38)
\end{align*}
\]

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In the table above an X shown that term is significant in the limit $\text{Re}_* \to \infty$ with $\eta$ fixed and $\alpha$ in the indicated range. The outer region is free of the viscous term, the inner is missing the last term, and the intermediate limit has only terms common to both regions. This is exactly the behavior one would expect for a two layer problem (Hinch(1991) p. 62.). Terms with ... would contribute to higher order terms in the Poincaré expansions.

However, because the turbulence problem in not closed, the governing relation for $\overline{uv}(y)$ is unknown, one cannot definitely conclude that an intermediate layer does not exist. Experimental and numerical data may never be precise enough to settle this question. On the other hand if two layers work, three are not needed.

Summary
\begin{itemize}
  \item $\text{Re}_*$ behavior of max. $\overline{uv}$ is explained by a composite expansion of a two-layer model and does not indicate that a third layer is required.
  \item Switch over of dominant terms in the momentum equation for various intermediate variable limits is as one would expect in a two-layer model.
  \item Because the turbulence equations are not closed, the mathematical possibility of a third layer always exists.
\end{itemize}

4. Defect law scale $U_0 - U_{\text{ave}}$

Next, let us turn to the question of the outer scaling proposed by Zagarola and Smits(1997). They found that a better correlation of the outer flow defect function $F_1(Y)$ (Eq. 6) was obtained if $u_*$ is replaced by $U_0 - U_{\text{ave}}$. In essence this is changing the gauge function in Eq. (2) from

$$\Delta_1 = \frac{u_*}{U_0} = \frac{1}{\kappa \ln (\text{Re}_*) + C_i - C_0}$$

(39)

to

$$\Delta_1 = \frac{U_0 - U_{\text{ave}}}{U_0}$$

(40)

Gauge functions in a Poincaré expansion are not unique. Indeed Kaplan(1957) in the solution of two-dimensional Stokes flow (a problem that also has a log overlap region) modified the gauge function and obtain a sharper numerical result. He changed

$$\frac{1}{\ln (\text{Re})} \text{ to } \frac{1}{\ln (\text{Re}) + C}$$

(41)

Van Dyke (1975, p243) calls this procedure telescoping.

By assuming the velocity profile one can compute the ratio of the Zagarola-Smits gauge function to the usual $\Delta_1 = u_*/U_0$. The velocity profile assumed was the law of the wall plus Coles' Wake law (including a corner correction of Lewkowicz(1982)). Integration yields

$$\frac{U_0 - U_{\text{ave}}}{u_*} = -C_0 + \frac{3}{2\kappa} \frac{3\Pi}{5\kappa} + \left[ 540.6 - 145/\kappa - 50 C_i \right] \frac{1}{\text{Re}_*}$$

$$= 4.46 + 95 \frac{1}{\text{Re}_*} - C_1 + \frac{C_2}{\text{Re}_*}$$

(42)

Hence, the two gauge functions are asymptotically equal. Zagarola-Smits have made an empirical improvement to the gauge function. The original form
\[ \Delta_1 = \frac{u_*}{U_0} \sim \frac{C_3}{\ln(Re_*) + C_4} \]  

(43)

has been replaced by

\[ \Delta_1 = \frac{u_*}{U_0} \cdot \frac{U_0 - U_{ave}}{u_*} \sim \frac{C_3\left[C_1 + \frac{C_2}{Re_*}\right]}{[\ln(Re_*) + C_4]} \sim \frac{C_5}{[\ln(Re_*) + C_6 - \frac{C_7}{Re_*} + ...]} \] 

(44)

This is equivalent to adding a higher order term as in the telescoping procedure.

Summary

• The use of \( U_0 - U_{\text{average}} \) to scale the defect law is the equivalent to adding a term of \( 1/Re_* \) to the gauge function. Applied mathematicians call this telescoping.

5. Value of \( \kappa \) and beginning of the log law

For the Stanford symposium in 1968 Coles analyzed the available boundary layer turbulence data by a new procedure. He assumed \( \kappa = 0.41 \) and \( C_0 = 5 \) and then essentially defined \( \delta \) and \( \Pi \) by the data fitting method. Anyone who has fitted data knows that four adjustable constants allows too much freedom. The Cole's method and the value \( \kappa = 0.41 \) were widely accepted. A serious challenge occurred when pipe data of Zagarola and Smits(1997) yielded \( \kappa = 0.44 \). In addition they indicated that the log law did not actually begin until \( y^+ > 500 \). Although the data are of high quality they require a correction for the Pitot tube employed. The correction used was quite old (Patel), disagrees with the other old possible correction (McMillan). At this time the proper Pitot probe correction is not known. Recent experiments of Li et al.(1999) did not resolve the question.

The latest high quality set of boundary layer measurements are from Osterlund et al.(2000). They find that \( \kappa = 0.38 \) and show that the log region begins at \( y^+ > 200 \). They conclude that by fitting closer to the wall previous researchers have obtained spuriously high values of \( \kappa \).

Summary

Neither of these results conflict with the classic theory, however, at some high Reynolds number one would expect that pipe and boundary layer inner regions would be the same.

6. Longitudinal Turbulence Fluctuations

As noted by Gad el Hak and Bandyophadhyay(1994) many workers have observed that the longitudinal turbulent fluctuations \( \bar{u^2}(y) / u_*^2 \) as a function of either \( y^+ \) or \( Y \) are not quite independent of \( Re_* \). Klewicki and Metzger (1996) in atmospheric tests on the Utah desert, Osterlund and Johansson (1999), and DeGraff and Eaton(1999) in detailed wind tunnel tests are among those to measure this effect. From a physical standpoint the idea (Townsend) is that large scale eddies can effect the flow very near the wall. Outer layer eddies are restrained in the \( y \) direction by the wall. Very near the wall the primary velocity of a large eddy is a \( u \) component. As the Reynolds number increases the outer layer is larger and contains a richer spectrum of eddies whose influence accumulates at the lower levels.

Marusic, et al.(1997) review data and construct a model with attached eddies as the central theme. The model is valid for \( y^+ > 100 \). They have a viscous effect that is a function of both inner and outer variables. The nature of the model leads them to speculate that inner variable similarity of \( \bar{u^2}(y) / u_*^2 \) in the viscous and buffer regions would not exist.
Fernholz and Finley (1996) reviewed data and noted the possibility of a Re effect in inner variables. They also noted that measurement errors could also be the cause. Mochizuki and Nieuwstadt (1996) were more positively on the side of wall variable scaling in inner region. Recent experiments on a thick boundary layer with a long run-up have been performed at École Centrale de Lille. The preliminary indications are that $\overline{u^2(y)}/u_*^2$ does scale on inner and outer variables as one would expect in a two layer theory (private communication Prof. M. Stanislas).

Considering the experimental difficulties in measuring turbulent statistics and the ease with which tripping and facility effects affect the outer layer, it is understandable that we have this controversy.

Summary

- The $\overline{u^2(y)}/u_*^2$ scaling problem (w also) exists in some experimental data, but is absent is other data. The physical reason for the lack of inner scaling is plausible. A complete explanation is needed. More experiments and DNS at higher Re would help. The turbulence community will probably not reach a conscience on this issue soon. The final explanation might even involve a three layer structure.

References