Topology Design of Compliant Cellular Structures with Contact-Enabled, Graded Stiffness

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A topology design method is presented for customizing cellular or lightweight structures with contact-enabled, graded stiffness. Structures with graded structural stiffness exhibit increasing stiffness with the magnitude of applied structural loads. In this work, stiffening is achieved via internal contact within the cellular structure. A continuum-based topology optimization approach is presented for tailoring cellular topologies with customized stiffness profiles that can be achieved only with internal contact. As part of the approach, contact behavior is modeled with an exponential function that gradually stiffens individual elements as contact begins to occur. Nonlinear finite element analysis is performed with a Newton Raphson procedure. The design problem is formulated to achieve targeted stiffness profiles at specified locations in the cellular structure. The resulting approach is capable of designing cellular topology with freeform, internal contact surfaces that emerge as the topology evolves.

I. Introduction

CELLULAR or honeycomb structures are known for their ultra-light load bearing and energy absorption properties. By customizing the cellular topology of these materials, it is possible to customize their multifunctional properties, including structural, thermal, and acoustic characteristics. Furthermore, solid freeform fabrication and other manufacturing processes provide capabilities for fabricating cellular structures with increasingly complex, freeform geometries. In this paper, we consider the design of lightweight structures with graded structural stiffness that increases with the magnitude of applied structural loading, as shown in Fig. 1.

There are several potential applications for cellular structures with graded stiffness. For example, the cushion of a pilot’s seat could exhibit relatively low stiffness (high compliance) under the pilot’s self-weight but gradually stiffen with additional loading. The high stiffness under additional loading could help the seat conform to the pilot’s body and position him during high-g maneuvers. A car bumper with graded stiffness could elastically deform and absorb most of the energy from a slow-moving impact, such as a rolling shopping cart, and then spring back to its original configuration upon release of the load. For loads of higher magnitude, the structure could stiffen gradually, providing a rigid surface to prevent further deformation of the vehicle. Similarly, prosthetic devices (such as a prosthetic foot) could be designed to provide low-stiffness, cushion-like support that gradually stiffens to provide stability as the supported load increases during walking.

In this work, the research objective is to devise a methodology for designing cellular structures with contact-enabled graded stiffness. To achieve graded stiffness, internal surfaces contact one another and gradually stiffen the structure as structural loads increase. The feasibility of the concept of contact-enabled stiffening was studied for the sample cell geometry shown in Fig. 2.¹ As the ANSYS finite element results indicate, horizontal cell walls contact one another as the magnitude of the vertical load increases, and the vertical walls become rigid column supports. The geometry does provide graded stiffness but offers little design flexibility. While the dimensions of the structure can be varied to adjust the stiffness profile, the topology of the structure remains unchanged. In this paper a topology design technique is proposed that tailors material distribution to achieve a desired stiffness profile.

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Topography design problems are usually formulated to optimize material distribution and attain high stiffness or a specified stiffness. In contrast, the optimization problem here is to design structures that gradually stiffen to match a desired stiffness profile. The approach of using contact conditions to enable graded stiffness between evolving surfaces makes the problem more complex. Existing work by Ananthasuresh and coauthors on designing contact-aided compliant mechanisms (CCM) involves topology optimization with contact surfaces, but the contact points are pre-specified. A specially formulated stiffness function gradually stiffens the elements as contact occurs within the structure, and truss-based topology optimization techniques are used. In this work, we utilize the SIMP (Solid Isotropic Material Penalization) approach, a continuum-based topology optimization technique, to generate freeform geometries with contact-enabled graded stiffness. The concept of modeling freeform, internal contact behavior in continuum-based topology optimization techniques is unique to this work; specifically, contact surfaces are not pre-specified but permitted to emerge as the cellular topology evolves.

The following section describes the topology design methodology with a detailed description of the finite element formulation and the optimization approach. In Section 3, the proposed methodology is demonstrated on two examples.

II. Methodology

An overview of the topology design methodology is provided in Fig. 3, in which the objective is to design the topology of a rectangular segment of cellular structure for desired graded stiffness under increasing compressive load. In brief, the methodology consists of the following steps:

- **Step 1.** Identify the material domain, loads, and boundary conditions.
- **Step 2.** Discretize material into finite elements and assign equal densities to each element.
- **Step 3.** Incorporate the contact stiffness function and control element overlap.
- **Step 4.** Solve the nonlinear finite element problem using Newton Raphson’s approximate method.
- **Step 5.** Calculate the objective function value and adjust its sensitivity using a density filter.
- **Step 6.** Updated element density variables using an optimization algorithm.
- **Step 7.** Check for convergence; if not, perform steps 3-6 until the optimal topology is obtained.
- **Step 8.** Convert optimized topology into a cleaner CAD representation.

The method is described in greater detail in the rest of this section. Specifically, the material is discretized into elements, and the finite element problem for contact is formulated as discussed in Sections A and B. The optimization algorithm and the objective function are introduced in Section C, followed by a sensitivity filter approach discussed in Section D.
A. Contact-Enabled SIMP Approach

In the proposed method, gradual stiffening behavior is achieved through contact within the structure. The structure is designed such that internal surfaces contact under increasing compressive loads, thereby drastically stiffening it. Among the various topology optimization approaches available, a continuum-based approach is better suited here for modeling contact between elements and enabling the design of freeform geometries. Here, we utilize the SIMP approach which modifies density variables assigned to each cell in a discretization of the design domain.\(^5\)

Each cell is represented as a four node, linear quadrilateral element in a plain-strain finite element formulation. The density (\(\rho\)) of each element directly affects the stiffness of that element as in Eq.1, where \(k_{a}\) is the stiffness matrix for a solid isotropic material and \(\rho\) is a constant that penalizes intermediate material densities and encourages convergence to structures with regions of full density (\(\rho \approx 1\)) and void (\(\rho \approx \rho_{\text{min}}\)):

\[
k_{a} = \rho^{n} k_{0}
\]
The SIMP algorithm is typically used to design minimally compliant structures; here, it is modified to yield structures with pre-specified, graded stiffness profiles via internal contact. Contact is not enabled in standard applications of the SIMP approach. Accordingly, elements with minimal density that are sandwiched between contacting surfaces could become distorted or inverted, particularly when the magnitude of element compression in Fig. 4 approaches or exceeds the initial height of the element, $\delta_0$. We therefore need a stiffening function that identifies the switch from no-contact to contact condition, prevents further deformation of the element, and enables transfer of structural loads between contacting solid surfaces. Ideally, the contact stiffness should be a step function that stiffens the element as soon as an element compresses completely. For a gradient based optimization program, however, it is essential to have a differentiable stiffness function. Saxena and Ananthasuresh have suggested an exponential stiffness function that closely approximates a step function, as illustrated in Eq. 2 and Fig. 4. In their work, they suggest using this formulation to model contact in beam elements in compliant mechanisms.

$$f_c(z) = \frac{\gamma}{1 + e^{-m \left(\frac{z}{\delta_0} - 1\right)}}$$  \hspace{1cm} (2)

where, $\delta_0$ is the initial height of the element, $z$ is the compressed distance in Fig. 4, and constant parameters $m$ and $\gamma$ determine the slope and magnitude of contact stiffness, respectively.

It is clear from Fig. 5 that the contact function closely represents a step function as the parameter $m$ is increased. As $m$ approaches infinity, it perfectly models the step function. Under low compressive loads, the displacement term $z$ is quite small compared to $\delta_0$ and therefore the contact stiffness is nearly zero. As the compression distance reaches $\delta_0$, the function tends to $\gamma$, which is much greater than the original stiffness.

The above stiffness function was originally proposed for beam elements and therefore is a function of the element length. For an irregular quadrilateral element the contact stiffness is a function of the relative compressed distance between adjacent nodes. Stiffness is calculated independently along the height and width of the element with the assumption that the elements are nearly rectangular throughout the analysis.
In Fig. 6, the contact stiffness corresponding to node 1 in the vertical direction is a function of the relative vertical distance of node 1 from node 4 in the deformed configuration \((z_{1y})\). The compression distance is calculated as

\[
z_{1x} = z_{2x} = U_{1x} - U_{2x}
\]

\[(3)\]

\[
z_{1y} = z_{4y} = U_{4y} - U_{1y}
\]

\[(4)\]

\[
z_{3x} = z_{4x} = U_{4x} - U_{3x}
\]

\[(5)\]

\[
z_{2y} = z_{3y} = U_{3y} - U_{2y}
\]

\[(6)\]

where \(U_{ix}\) and \(U_{iy}\) are the nodal displacement of the \(i\)th node in the x and y directions respectively. As the compression distances are relative distances between adjacent nodes, \(z_{1y}\) and \(z_{4y}\) are equal.

The contact stiffness matrix \((k_c)\) for an element is populated as in Eq. 7 where \(f_c(z)\) refers to the contact function introduced in Eq. 2:

\[
f_{c}(z) = \begin{cases} 0, & z \\ f_c, & z \geq z \end{cases}
\]

where \(f_c\) is the contact force and \(z\) is the compression distance.

Figure 5. Contact model in a compliant mechanism problem

Figure 6. Calculation of compressed height for different nodes
The effective local stiffness \( k_e \) of an element is defined as the sum of its contact stiffness matrix \( k_c \) and its local stiffness matrix \( k_i \) which is a function of its density, according to Eqs. 1, 8 and 9. The individual local matrices are then assembled together into the global stiffness matrix \( K \), as shown in Eq. 10.

\[
k_e = \rho_i^2 k_0 \quad \text{(8)}
\]

\[
k_e(\rho_i) = k_i + k_c \quad \text{(9)}
\]

\[
K = \sum_{i=1}^{N} k_e(\rho_i) \quad \text{(10)}
\]

**B. Modified Newton Raphson Method**

Given a set of boundary conditions, a typical finite element problem is solved in a single step for the nodal displacements in a structure. For a contact problem in which structural stiffness changes with displacement, the finite element problem is solved iteratively with the Newton Raphson method. The stiffness matrix is updated at every step as the structure deforms.

For a structure with non-linear stiffness as shown in Fig. 1, the constitutive equation for a given external load \( F_{ext} \) is written as follows:

\[
\int K \cdot U = F_{ext} \quad \text{(11)}
\]

This nonlinear problem is solved by dividing the total external load into load steps and iteratively solving for convergence within each load step. The stiffness matrix is recalculated in every iteration and is assumed to be linear. For the first iteration in the first load step, the stiffness matrix \( K_1 \) is determined from the undeformed configuration of the structure. For an initial load step of \( F_1 \), the nodal displacement is calculated linearly as follows:

\[
dU_1 = K_1^{-1} F_1 \quad \text{(12)}
\]

where superscripts represent load steps, and subscripts represent iterations within load steps. In the calculation of the stiffness matrix, the stiffness of elements with nearly zero density is heavily discounted, and they effectively behave as void elements. Such elements can potentially invert upon loading, and it is essential to account for contact stiffening for these elements. The contact condition is formulated as follows:

\[
\delta_i = 0 \quad \text{if} \quad z(i, j) \leq \delta_0 = 1, \quad \text{then contact has not occurred between nodes} \quad i \quad \text{and} \quad j \quad \text{for all} \quad i, j, \quad \text{and} \quad

U_n = U_{n-1} + dU_n \quad \text{(13)}
\]

The incremental nodal displacement \( z \) is calculated for the displacement vector \( U_{n-1} \) from Eq. 3 – Eq. 6.

If the contact condition is not met, the computed displacement \( (U_n) \) directly determines the new, deformed geometry. For example if there are no overlapping elements at the first increment, the total displacement \( (U_1) \) of the structure is calculated by adding the displacement increment \( (dU_1) \) to the initial displacement \( (U_0) \). Usually \( U_0 \) is equal to zero as the analysis starts with the undeformed configuration.
If the contact condition is not satisfied, then contact has occurred, and large nodal displacements have led to element inversion. To ensure that contact stiffness is gradually incorporated, the incremental displacement vector \((dU_n^i)\) is scaled down. To determine the scaling factor, we first determine the maximum compression within any element. For the example row of elements shown in Fig. 7, assume that displacements are fixed for the bottom row of evenly numbered nodes 2 – 14. The maximum compression, \(h_{\text{max}}\), is then in the vertical direction between nodes 9 and 10, as that is the maximum displacement (x or y direction) from increment \(n-1\) to \(n\) among all elements.

![Figure 7. Calculating scaling factor \(\lambda\) from deformation at increments \(n\) and \(n-1\) for nodes 1 – 14](image)

For a generic scenario, the maximum compression, \(h_{\text{max}}\), is the maximum difference in incremental displacement of adjacent nodes from increment \(n-1\) to \(n\) throughout the structure (Eq. 14).

\[
\text{if } dU_n^i(i) - dU_n^i(j) \geq h_{\text{max}} \\
\text{then } h_{\text{max}} = dU_n^i(i) - dU_n^i(j)
\]

(Eq. 14)

The scaling factor \(\lambda\) is then the ratio of the compressed height of the corresponding element in the \(n-1\)th iteration to \(h_{\text{max}}\) (Eq. 15):

\[
\lambda = \alpha \cdot \frac{1 - z(i, j)}{h_{\text{min}}}
\]

(Eq. 15)

where \(\alpha (= 0.5)\) is the scalar step length. In the above example, the scaling factor would be the ratio between the height of the element corresponding to \(z_{n-1}(9,10)\) and \(h_{\text{max}}\). After the scaling factor is determined, the total displacement in the \(n\)th iteration is calculated as in Eq. 16. Figure 8 illustrates the scaling of the incremental displacement vector and the displacement vector calculation.

\[
U_n^i = U_{n-1}^i + \lambda \cdot dU_n^i
\]

\[
z_n(i, j) = z_{n-1}(i, j) + \lambda \cdot (dU_n^i(i) - dU_n^i(j))
\]

(Eq. 16)

After the nodal displacement is updated, the traditional Newton Raphson method is continued for the next increment. At every increment, the measured external force is calculated (Eq. 17).

\[
F_{n-\text{corr}}^i = \int K(U) dU
\]

\[
= \int \sum_{i=1}^{N} k_i(U) dU + \sum_{i=1}^{N} k_i \cdot U_n^i
\]

(Eq. 17)

Convergence of the Newton-Raphson’s method at each step is determined by the error between the applied external force and measured force, \(Error_n^i\) calculated as in Eq. 18.

\[
Error_n^i = F^i - F_{n-\text{corr}}^i
\]

if \(\text{norm}(Error_n^i) > \varepsilon\), the solution has not converged

(Eq. 18)

If the error is within a small tolerance value, e.g., \(\varepsilon = 10^{-3}.F^i\), the solution is said to have converged. If this condition is not satisfied, the next iteration is performed starting from Eq. 12. The nodal displacement in the \(n\)th
The calculated error in forces $Error_n$ is the new applied external force $F_{n+1}$.

The procedure has been described for a single load step. As previously mentioned, the total external force is divided into multiple load steps, and this procedure is repeated for each load step independently. Within each load step, force convergence is achieved, and the converged displacements define the initial geometry for the next load step. For example, if $U_{n+1}$ is the converged displacement for the $1st$ load step, the second load step starts with this deformed geometry as its initial state ($U_0^2$) (Eq. 19).

$$U_0^2 = U_1^1$$

It is advantageous to increase the number of load steps to capture drastic changes in the stiffness profile and to increase the accuracy of the solutions, but increasing the number of load steps also increases the total number of iterations and the computational time. For the example problems considered in this paper, the total external load is divided into 10 load steps. Within each load step, the number of iterations is limited to a maximum of 500, beyond which a diverging solution is reported.

**C. Optimization Problem Formulation**

The optimization problem is formulated as a weighted function of the squared difference between observed displacements and target displacements. As illustrated in Figure 9, target displacements are specified at selected nodes on the top surface of the structure and for multiple load levels such that the structure stiffens with increasing load. The objective function is formulated as follows for multiple loads ($F_1, F_2, \text{etc.}$):

Minimize: $$f = w_1 \left[ 1 - \frac{U_1}{U_{1\text{Target}} / F_i} \right]^2 + w_2 \left[ 1 - \frac{U_2}{U_{2\text{Target}} / F_i} \right]^2 + \ldots$$

Constraint: $$\frac{1}{n} \sum_{i=1}^{n} \left[ 1 - \frac{(\rho_i - 0.5)^2}{0.25} \right] < 0.2$$
where, $w_i$ is the weight assigned to load $F_i$ and $U_j$ and $U_{j\text{Target}}$ are the observed and target nodal displacements at pre-specified node $j$.

Most topology optimization problems are formulated to achieve minimum compliance with a limited amount of material. Such problems are bounded by a volume fraction constraint so that there is a tradeoff between compliance and volume. In contrast, the present challenge is to design structures with targeted stiffness profile, such that the structures are compliant initially and gradually stiffen under increasing compressive loads. In these contexts, a volume fraction constraint would be ineffective because a fully dense structure could not achieve the targeted stiffness profile. However, a constraint (Eq. 20) is added to the formulation to penalize elements with intermediate densities and encourage convergence to fully dense (solid) and minimally dense (void) elements. The proposed constraint in Eq. 20 is a parabolic function of the element density. This effective volume constraint assigns a maximum penalty to elements with a density of 0.5 and parabolically decreasing penalties for higher or lower densities, converging to zero for element densities of 0 or 1. The effective volume of the structure is fixed at an upper limit of 0.2. This upper limit corresponds to the effective volume calculated if the whole structure were effectively solid with density of 0.95 or effectively void with density of 0.05. This limit is very aggressive but effective in eliminating intermediate density elements that are physically unrealizable.

The method of moving asymptotes (MMA) is used to solve the optimization problem. It is a gradient based technique and is capable of varying the rate of convergence when needed. For the objective function mentioned above, the gradient function is represented in Eq. 21.

\[
\frac{\partial f}{\partial \rho_i} = -2w_i \left[ \left( 1 - \frac{U_i}{U_{1\text{Target}}} \right) \frac{1}{F_1} \frac{\partial U_i}{\partial \rho_i} + \left( 1 - \frac{U_2}{U_{2\text{Target}}} \right) \frac{1}{F_2} \frac{\partial U_2}{\partial \rho_i} + \ldots \right]
\]

\[
-2w_i \left[ \left( 1 - \frac{U_i}{U_{1\text{Target}}} \right) \frac{1}{F_2} \frac{\partial U_i}{\partial \rho_i} + \left( 1 - \frac{U_2}{U_{2\text{Target}}} \right) \frac{1}{F_2} \frac{\partial U_2}{\partial \rho_i} + \ldots \right] + \ldots
\]

As shown in the equation, the sensitivity of the objective function with respect to each density variable is a function of the sensitivity of the nodal displacement vector. The sensitivity of the nodal displacement vector with respect to the density variable is derived from a force balance as follows:

\[
F_{\text{external}} = F_{\text{internal}} + F_{\text{contact}}
\]

\[
F_{\text{external}} = \sum_{i=1}^{N} k_i \cdot U + F_{\text{contact}}
\]

In Eq. 23, the contact force is related to contact stiffness as follows:

\[
k_c = \frac{dF_{\text{contact}}}{dU}
\]

Differentiating Eq. 23 with respect to $\rho_i$ yields the following relationship:

\[
\frac{\partial F_{\text{external}}}{\partial \rho_i} = \sum_{j=1}^{N} 3 \rho_j^2 k_0 \delta_{ij} \cdot U + \sum_{j=1}^{N} k_j \frac{\partial U}{\partial \rho_i} + \frac{\partial F_{\text{contact}}}{\partial U} \frac{\partial U}{\partial \rho_i}
\]

such that,
\[ \delta_{ij} = 1, \text{ if } i = j \]
\[ = 0, \text{ if } i \neq j \]

\( F_{\text{external}} \) is a constant, therefore its derivative is zero.
\[ \frac{\partial F_{\text{external}}}{\partial \rho_i} = 0 \]  

Also from Eq. 10 and Eq. 24 we have,
\[ K = \sum_{j=1}^{N} k_j + \frac{\partial F_{\text{contact}}}{\partial U} \]  

Substituting Eqs. 26 and 27 into Eq. 25 yields:
\[ \sum_{j=1}^{N} 3\rho_j^3k_0\delta_{ij} \cdot U + K \cdot \frac{\partial U}{\partial \rho_i} = 0 \]
\[ \frac{\partial U}{\partial \rho_i} = -K^{-1} \sum_{j=1}^{N} 3\rho_j^3k_0\delta_{ij} \cdot U \]  

The gradient of the objective function is utilized by the optimization algorithm to steer the density variables towards the optimal topology. At every optimization iteration, the value of the minimization function is compared against \( \varepsilon \) (= 10E-3), a very small scalar value. The algorithm is assumed to converge if the function is less than \( \varepsilon \). The algorithm is terminated if this condition is not satisfied within 125 iterations, and the iteration with the smallest functional value is declared as optimal.

The topologies generated from such optimization techniques can be quite intricate and difficult to manufacture using conventional techniques. The motivation in generating such structures is the availability of manufacturing processes using solid freeform fabrication (SFF)\(^9\) techniques. Manufacturing process such as selective laser sintering (SLS)\(^10\) can build almost any open or closed geometry and realize nearly any form of topology.

D. Filtering Technique

One of the issues with continuum based topology optimization techniques is the possibility of stray (disconnected) solid elements in the structures. Such checker board structures could meet the required objective quite well, but are not feasible designs from a manufacturing perspective. To ensure that the topologies are continuous, a density filter is incorporated that modifies individual element sensitivities as a function of surrounding element sensitivities.\(^8\) In the original SIMP code, Sigmund suggests having a density filter as shown in Eq. 29 that modifies the gradients of the objective function, \( f \), as follows:
\[ \frac{\partial f}{\partial \rho_i} = \frac{1}{\rho_i} \sum_{j=1}^{N} \left( d_{ij} - r \right) \sum_{j=1}^{N} \left( d_{ij} - r \right) \rho_j \frac{\partial f}{\partial \rho_j} \]  

The above filter modifies the function gradients with the product of densities and gradients of surrounding elements. When the surrounding elements are nearly void, they have negligible contribution to this term and do not cause any change in the topology. Similarly for a void element surrounded by a few solid elements, the full contribution of the solid element forces the void element to become more solid. This filter eventually turns most void elements to intermediate density elements, and the boundary between solid and void elements is not very distinct which is essential in a contact problem.

In an alternate filter suggested by Wang and Wang each surrounding element has full contribution to the gradient irrespective of its density.\(^12\) This filter modifies the density gradients as in Eq. 30.
\[ \frac{\partial f}{\partial \rho_i} = \sum_{j} \left( d_{ij} - r \right) \sum_{j} \left( d_{ij} - r \right) \frac{\partial f}{\partial \rho_j} \]  

where, \( r \) is the filter radius and \( d_{ij} \) is the distance between the centroid of element \( i \) and neighboring element \( j \) as shown in Fig. 10.
Filter is active when \( r > d_{ij} \)

**Figure 10. Region of influence of the filter on the sensitivity of the function**

With this filter, a solid element surrounded by void elements turns void and vice versa. The filter generates a crisper topology with fewer intermediate density elements. The factor in the filter that controls its extent of influence is the filter radius \( r \). The filter radius is the largest distance between centers of neighboring elements within which the filter has an effect. A radius of 1.5 is fixed for the present applications, based on a comparison of results with lower and higher filter radii.

### III. Results

The proposed topology design technique is demonstrated on two examples. For each example, the structure is assumed to have a uniform cross-section in the z direction throughout and the analysis is performed as a 2D finite element problem in the x-y plane, as illustrated in Fig. 11. The outer dimensions of the material are fixed at 90mm x 55mm x 25mm as shown in Fig. 11. The x-y section of the material is divided into a grid of 54x15 elements. To ensure that there is always a top surface to support the external force and a bottom surface to constrain the structure, the top and bottom rows of elements are fixed at unit density. A uniformly distributed compressive force is applied on the top surface in increments of 20kg, 40kg and 80 kg. For each force increment, target displacements are specified on the top surface at equally spaced nodes. Target displacements are specified such that the structure stiffens as it is loaded.

#### A. Example 1

Target displacements for the first example are illustrated in Fig. 12, with nodes 1 and 9 corresponding to the left and right sides of the structure in the x-y plane in Fig. 11. Iterations begin with each element assigned a density of 0.5 (Fig. 13). The optimization algorithm reallocates material to match the target displacements as closely as possible. Intermediate results in Fig. 13 illustrate the evolving topology. As shown, contact is enabled between the top surface and interior surfaces to match the target displacements more closely. By iteration 15, material starts to distribute towards the left side to achieve higher stiffness targets there. In further iterations, we see that significant material is located at the center of the top surface to provide adequate stiffness to the top surface and to facilitate contact with the bottom surface for stiffening under increasing compressive loads.

**Figure 11. Dimensions of the analyzed material with the expected deformation profile**
As shown in Fig. 12, the resulting topology matches the desired deformation profile quite closely and is intended to provide a starting point for further shape and size optimization. These post-processing steps would alleviate some of the features in the converged topology that are not physically realizable. For example, the converged topology is not completely connected between the top and bottom surface. This feature appears because the target displacements require significant displacements across the top surface, and it could be avoided by specifying near zero displacements at the ends. A few grey (intermediate density) elements are also visible on the left side; these elements provide a rigid support under higher compressive load, while the white cells have totally collapsed. Further refinement of the topology would require these elements to be converted to solid or removed entirely.

![Figure 12. Displacement targets and actual displacements for the optimal structure in Example 1](image)

**B. Example 2**

In the second example, the target displacements are much larger with more drastic variations in the profile, as illustrated in Fig. 14. As in the earlier example, the iterations start with each element in the topology assigned a density of 0.5 (Fig. 15). In the initial iterations a prominent chunk of material is located at the center that supports the structure from excessive deformation. Grey elements on the sides start to mature into solid and void cells and vertical features develop by iteration 25. These vertical columns support the material on the sides as the target displacements there are nearly zero. On the right end, the displacement gradient is quite gradual, and a truss-like structure develops to provide further stiffness in the 50th iteration. By the 70th iteration the topology is almost refined with fewer grey (intermediate density) elements, which are almost eliminated by the 125th iteration.

As mention in the previous example, the topology generated from this technique would require post-processing to refine the dimensions and shapes of the structure before it is manufacturable. Parametric optimization would also help in refining the right-side truss structure that is not fully developed.

When actual displacements of the final structure are compared with target displacements, it is quite clear that the targeted displacements are significantly under-achieved, because the target displacements are overly ambitious and unachievable in the specified domain. The target displacements at the center of the structure are too large, for example. The algorithm does match the desired deformed profile as closely as possible, however, by minimizing the objective function. The example also shows the capability of the algorithm in generating internal structures to provide contact surfaces that stiffen it upon further compression.
IV. Conclusion

The primary objective of this paper is to establish a method for topology design of compliant cellular structures with graded stiffness achieved through internal surface contact. The design problem is to generate surfaces within the structure that contact one another under increasing loads and stiffen the structure. Graded stiffness has been achieved in the literature using multiphase materials, but the idea of strategically enforcing contact to achieve a desired stiffening profile during a continuum-based topology optimization process is unique to the work presented in this paper.

The topology optimization technique of SIMP is modified to incorporate contact behavior via a nonlinear contact stiffness function. The contact stiffness is approximated by an exponential function that makes the problem continuous and solvable using gradient-based optimization techniques. The contact-enabled SIMP approach generates potential contacting surfaces iteratively as the structure is loaded and updates the geometry as required. Since the contact stiffness is a nonlinear function, iterative finite element techniques are used to solve the problem.

The design methodology is applied to two example problems. In the examples, a specific external deformation profile is desired in which a structure stiffens under increasing compressive load. The desired stiffness profile is achieved by strategically placing material within the structure such that stiffening behavior is achieved through surface contact. The examples demonstrate the capability of this methodology to achieve a target graded stiffness profile and to introduce contact surfaces when stiffening is required.

The methodology has been validated with several design problems, but several aspects of the approach can be improved. One obvious opportunity is to refine the approach to generate crisper topologies. Currently, the generated topologies are not refined to the extent that there is a distinct boundary between solid and void elements, which is
Figure 14. Displacement targets and actual displacements for the optimal structure in Example 2

Figure 15. Undeformed and deformed state of the Topology as it develops for Example 2
essential to ensure that contact behavior is modeled well. In the problem formulation, the dense elements are penalized severely to eliminate them, but there are still some grey elements in the final topology and additional work needs to be done in this respect. Also, some of the generated topologies, solid elements are connected at single vertices. Such topologies are not physically feasible and tend to be stiffer than expected when approximated as continuous structures. Modifications of the density filter function and its influence region may be required. On the other hand, it is important to recognize that topologies generated with this methodology need to be refined to eliminate these features and intermediate density elements before they are represented in CAD and fabricated. Finally, the work presented here is an initiative towards modeling contact behavior in cell-based topology optimization problems. The approach is not limited to graded stiffness applications but can also be applied to other design problems, such as compliant mechanism design, in which surface contact is essential to transmit energy.

References