Review Problems

Group A

1. Find all solutions to the following linear system:

   \[ x_1 + x_2 = 2 \]
   \[ x_2 + x_3 = 3 \]
   \[ x_1 + 2x_2 + x_3 = 5 \]

2. Find the inverse of the matrix \[
\begin{bmatrix}
0 & 3 \\
2 & 1
\end{bmatrix}
\]

3. Each year, 20% of all untenured State University faculty become tenured, 5% quit, and 75% remain untenured. Each year, 90% of all tenured S.U. faculty remain tenured and 10% quit. Let \( U_t \) be the number of untenured S.U. faculty at the beginning of year \( t \), and \( T_t \) the tenured number.

Use matrix multiplication to relate the vector \[
\begin{bmatrix}
U_{t+1} \\
T_{t+1}
\end{bmatrix}
\]
to the vector \[
\begin{bmatrix}
U_t \\
T_t
\end{bmatrix}
\].

4. Use the Gauss-Jordan method to determine all solutions to the following linear system:

   \[ 2x_1 + 3x_2 = 3 \]
   \[ x_1 + x_2 = 1 \]
   \[ x_1 + 2x_2 = 2 \]

5. Find the inverse of the matrix \[
\begin{bmatrix}
0 & 2 \\
1 & 3
\end{bmatrix}
\].

6. The grades of two students during their last semester at S.U. are shown in Table 2.

Courses 1 and 2 are four-credit courses, and courses 3 and 4 are three-credit courses. Let GPA\(_i\) be the semester grade point average for student \( i \). Use matrix multiplication to express the vector \[
\begin{bmatrix}
\text{GPA}_1 \\
\text{GPA}_2
\end{bmatrix}
\]
in terms of the information given in the problem.

7. Use the Gauss-Jordan method to find all solutions to the following linear system:

   \[ 2x_1 + x_2 = 3 \]
   \[ 3x_1 + x_2 = 4 \]
   \[ x_1 - x_2 = 0 \]

8. Find the inverse of the matrix \[
\begin{bmatrix}
2 & 3 \\
3 & 5
\end{bmatrix}
\].

9. Let \( C_t \) = number of children in Indiana at the beginning of year \( t \), and \( A_t \) = number of adults in Indiana at the beginning of year \( t \). During any given year, 5% of all children become adults, and 1% of all children die. Also, during any given year, 3% of all adults die. Use matrix multiplication to express the vector \[
\begin{bmatrix}
C_{t+1} \\
A_{t+1}
\end{bmatrix}
\]
in terms of \[
\begin{bmatrix}
C_t \\
A_t
\end{bmatrix}
\].

10. Use the Gauss-Jordan method to find all solutions to the following linear equation system:

   \[ x_1 - x_3 = 4 \]
   \[ x_2 + x_3 = 2 \]
   \[ x_1 + x_2 = 5 \]
11 Use the Gauss-Jordan method to find the inverse of the matrix
\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

12 During any given year, 10% of all rural residents move to the city, and 20% of all city residents move to a rural area (all other people stay put!). Let \( R_t \) be the number of rural residents at the beginning of year \( t \), and \( C_t \) be the number of city residents at the beginning of year \( t \). Use matrix multiplication to relate the vector \( \begin{bmatrix} R_{t+1} \\ C_{t+1} \end{bmatrix} \) to the vector \( \begin{bmatrix} R_t \\ C_t \end{bmatrix} \).

13 Determine whether the set \( V = \{ [1 2 1], [2 0 0] \} \) is a linearly independent set of vectors.

14 Determine whether the set \( V = \{ [1 0 0], [0 1 0], [-1 -1 0] \} \) is a linearly independent set of vectors.

15 Let \( A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \).

a. For what values of \( a, b, c, \) and \( d \) will \( A^{-1} \) exist?

b. If \( A^{-1} \) exists, find it.

16 Show that the following linear system has an infinite number of solutions:
\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
2 \\
3 \\
4 \\
1
\end{bmatrix}
\]

17 Before paying employee bonuses and state and federal taxes, a company earns profits of $60,000. The company pays employees a bonus equal to 5% of after-tax profits. State tax is 5% of profits (after bonuses are paid). Finally, federal tax is 40% of profits (after bonuses and state tax are paid). Determine a linear equation system to find the amounts paid in bonuses, state tax, and federal tax.

18 Find the determinant of the matrix \( A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \).

19 Show that any \( 2 \times 2 \) matrix \( A \) that does not have an inverse will have \( \det A = 0 \).

**Group B**

20 Let \( A \) be an \( m \times m \) matrix.

a. Show that if rank \( A = m \), then \( Ax = 0 \) has a unique solution. What is the unique solution?

b. Show that if rank \( A < m \), then \( Ax = 0 \) has an infinite number of solutions.

21 In our study of Markov chains (see Chapter 19), we will encounter the following linear system:
\[
[x_1 \ x_2 \ \cdots \ x_n] = [x_1 \ x_2 \ \cdots \ x_n]P
\]

where
\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}
\end{bmatrix}
\]

If the sum of each row of the \( P \) matrix equals 1, use Problem 20 to show that this linear system has an infinite number of solutions.

22 The national economy of Seriland manufactures three products: steel, cars, and machines. (1) To produce $1 of steel requires 30¢ of steel, 15¢ of cars, and 40¢ of machines. (2) To produce $1 of cars requires 45¢ of steel, 20¢ of cars, and 10¢ of machines. (3) To produce $1 of machines requires 40¢ of steel, 10¢ of cars, and 45¢ of machines. During the coming year, Seriland wants to consume \( d_c \) dollars of steel, \( d_c \) dollars of cars, and \( d_m \) dollars of machinery.

For the coming year, let
\[
s = \text{dollar value of steel produced} \\
c = \text{dollar value of cars produced} \\
m = \text{dollar value of machines produced}
\]

Define \( A \) to be the \( 3 \times 3 \) matrix whose \( ij \)th element is the dollar value of product \( i \) required to produce $1 of product \( j \) (steel = product 1, cars = product 2, machinery = product 3).

a. Determine \( A \).

b. Show that
\[
\begin{bmatrix}
s \\ c \\ m
\end{bmatrix} = \begin{bmatrix}
2 & 4 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
d_s \\ d_c \\ d_m
\end{bmatrix}
\]

(Hint: Observe that the value of next year's steel production = (next year's consumer steel demand) + (steel needed to make next year's steel) + (steel needed to make next year's cars) + (steel needed to make next year's machines). This should give you the general idea.)

c. Show that Equation (24) may be rewritten as
\[
(I - A) \begin{bmatrix}
s \\ c \\ m
\end{bmatrix} = \begin{bmatrix}
d_s \\ d_c \\ d_m
\end{bmatrix}
\]

d. Given values for \( d_s, d_c, \) and \( d_m \), describe how you can use \( (I - A)^{-1} \) to determine if Seriland can meet next year's consumer demand.

e. Suppose next year's demand for steel increases by $1. This will increase the value of the steel, cars, and machines that must be produced next year. In terms of \( (I - A)^{-1} \), determine the change in next year's production requirements.

\[\text{Based on Leontief (1966). See references at end of chapter.}\]