MIDTERM II - Solutions

1a. (10%) To determine the value of the basic variables, let

\[ \mathbf{x}_B = (x_2, x_3) = \mathbf{B}^{-1} \mathbf{b} = \frac{1}{-6} \begin{bmatrix} 5 & -14 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \]

1b. (10%) To determine the value of the dual variables, let

\[ \mathbf{\pi} = (\pi_1, \pi_2) = \mathbf{c}_b \mathbf{B}^{-1} = (-4, -7) \frac{1}{-6} \begin{bmatrix} 5 & -14 \\ 1 & -4 \end{bmatrix} = (9/2, -14) \]

1c. (10%) The fundamental law of duality says that in a primal-dual pair if one has a finite solution then so does the other and the two objective functions are equal.

\[ z_P = -4(2) - 7(2) = -22 \]
\[ z_D = 20(9/2) + 8(-14) = -22 \]

We must now determine if the dual problem is feasible.

Minimize \( w = 20\pi_1 + 8\pi_2 \)
subject to
\[ 5\pi_1 + \pi_2 \leq -3 \]
\[ -4\pi_1 - \pi_2 \leq -4 \]
\[ 14\pi_1 + 5\pi_2 \leq -7 \]
\[ -2\pi_1 - \pi_2 \leq 1 \]
\[ \pi_1 \text{ and } \pi_2 \text{ unrestricted} \]

Only need to check constraints 1 and 4; constraints 2 and 3 should be tight due to the complementary slackness condition.

Constraint 1: \( 5(9/2) - 14 = 22.5 - 14 = 7.5 > -3 \) so we do not have the optimal solution.
2a. (5%) The transportation array is as follows.

\[
\begin{array}{ccc|c}
 & 3 & 4 & \text{Supply} \\
1 & 2 & 3 & 4 \\
2 & 4 & 3 & 5 \\
\text{Demand} & 3 & 6 & \\
\end{array}
\]

First we note that \( \sum s_i = \sum d_j = 9 \) so problem is balanced. The NW corner rule gives \( x_{11} = 3, x_{12} = 1, x_{21} = 0, x_{22} = 5, z = 24 \).

2b. (20%) To test for optimality, we compute the dual variables and then the reduced costs.

\[
\begin{align*}
&x_{11} > 0 \Rightarrow u_1 + v_1 = 2 \\
&x_{12} > 0 \Rightarrow u_1 + v_2 = 3 \\
&x_{22} > 0 \Rightarrow u_2 + v_2 = 3
\end{align*}
\]

We can arbitrarily set one of the dual variables to 0. Let \( u_1 = 0 \). This leads to \( v_1 = 2, v_2 = 3 \) and \( u_2 = 0 \).

Now compute the reduced cost for the nonbasic cell (2,1):

\[
\begin{align*}
&c'_{21} = c_{21} - u_2 - v_1 = 4 - 0 - 2 = 2 > 0
\end{align*}
\]

Therefore, we have the optimal solution.

2c. (5%) The A-matrix of the transportation problem is totally unimodular. This means that all the vertices of the polyhedron that defines the feasible region are integral. Because any solution obtained with the simplex method will be a vertex, it is guaranteed to be integral as well.
3. (20%) This problem is known as a traveling salesman problem (TSP) and cannot be formulated as any of the network models presented in Chapters 5 and 6. It is a combination of a minimal spanning tree problem and an assignment problem, and can be formulated using a graph with \( n + 1 \) nodes, where the distance between nodes \( i \) and \( j \) is \( d_{ij} \). Let

\[
x_{ij} = 1 \text{ if relative } i \text{ is visited immediately before relative } j; \quad 0 \text{ otherwise.}
\]

\( N = \{0, 1, 2, \ldots, n\} \)

**Model**

Minimize \( z = \sum_{i=0}^{n} \sum_{j=0}^{n} d_{ij} x_{ij} \)

(exactly one successor for each node) \( \sum_{j=0}^{n} x_{ij} = 1, \quad i = 0, 1, \ldots, n \) \hspace{1cm} (C1)

(exactly one predecessor for each node) \( \sum_{i=0}^{n} x_{ij} = 1, \quad j = 0, 1, \ldots, n \) \hspace{1cm} (C2)

(subtour elimination) \( \sum_{j \in S} \sum_{j \in S} x_{ij} \leq |S| - 1, \quad S \subset N, \ 2 \leq |S| \leq n - 2 \) \hspace{1cm} (C3)

(integrality) \( x_{ij} = 0 \text{ or } 1, \quad i \neq j = 0, 1, \ldots, n \) \hspace{1cm} (C4)

This model is discussed in Chapter 7. The first two constraints (C1) and (C2) are from the assignment model; (C3) prevents cycles that do not include all the nodes and is known as a subtour elimination constraint.
4a. (10%) Let \( x_{ij} = 1 \) if arc \((i, j)\) is selected; 0 otherwise. Note that \( \Sigma s_i = \Sigma d_j = 90 \) so the problem is balanced and no dummy nodes need to be added.

The LP formulation is:

\[
\begin{align*}
\text{Minimize} & & 1x_{BA} + 4x_{AC} + 6x_{AD} + 2x_{BC} + 5x_{BE} + 5x_{CE} + 3x_{CD} \\
\text{subject to} & & x_{AD} + x_{AC} - x_{BA} = 50 & \text{(node A flow balance)} \\
& & x_{BA} + x_{BC} + x_{BE} = 80 & \text{(node B flow balance)} \\
& & x_{CD} + x_{CE} - x_{AC} - x_{BC} = 0 & \text{(node C flow balance)} \\
& & -x_{AD} - x_{CD} = -70 & \text{(node D flow balance)} \\
& & -x_{CE} - x_{BE} = -60 & \text{(node E flow balance)} \\
& & x_{BE} \leq 40, x_{AD} \leq 40 & \text{(capacity limits)} \\
& & x_{ij} \geq 0, \text{ for all nodes } i \text{ and } j & \text{(nonnegative flow)}
\end{align*}
\]

4b. (15%) The MSP is \( x_{BA} = 1, x_{BC} = 1, x_{CD} = 1, x_{BE} = 1 \) (or \( x_{CE} = 1 \)).

In the model, there are 5 flow balance constraints with one being redundant. This means that there will be 4 basic variables. The columns associated with the MST solution are

<table>
<thead>
<tr>
<th></th>
<th>( x_{BA} )</th>
<th>( x_{BC} )</th>
<th>( x_{CD} )</th>
<th>( x_{BE} )</th>
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<tbody>
<tr>
<td>0</td>
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<td>1</td>
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</tbody>
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After eliminating one row, say the first row, we are left with the following matrix:  

- 4 -
This matrix is upper triangular and hence nonsingular. This means that its rank is 4 so its columns are linearly independent and thus form a basis.