Benders Decomposition Homework

1. The Fixed-Charge Network Flow (FCNF) problem: We are given a directed network with a set of nodes $V$ (facilities) and a set of arcs $A$. An arc $e = (i,j)$ that points from node $i$ to node $j$ means that there is a direct shipping route from $i$ to $j$. Associated with each node $i$, is a demand $b_i$. Node $i$ is a demand, supply, or transit point depending on whether $b_i$ is positive, negative, or zero, respectively. We assume that the net demand is zero; that is, $\sum_{i \in V} b_i = 0$.

Each arc $(i,j)$ has a flow capacity $u_{ij}$ and a unit flow cost $h_{ij}$. Also, if an arc $(i,j)$ is used to carry flow a fixed cost of $c_{ij}$ is incurred. The integer programming formulation of the FCNF problem is given on page 229 of the text. Note that the flow costs $\sum_{ij \in A} h_{ij} x_{ij}$ are missing and should be added.

Develop a Benders reformulation for the FCNF problem. Discuss its advantages.

2. Consider the Uncapacitated Facility Location (UFL) problem given on page 10 of the text. Replace the “weak” formulation by the “strong” formulation given on page 123.

   a. Develop a Benders reformulation for this problem by first decomposing it into $m$ subproblems, one for each $i \in M$. What is the upper limit on the number of constraints in the master problem?

   b. Develop a second Benders reformulation without decomposing UFL into $m$ subproblems. Now how many constraints are there in the master problem?

3a. Solve the following mixed-integer program using Benders decomposition.

$$\begin{align*}
\text{max} \quad & 2x_1 + x_2 + 3x_3 + 7y_1 + 5y_2 \\
\text{subject to} \quad & 9x_1 + 4x_2 + 14x_3 + 35y_1 + 24y_2 \leq 80 \\
& -x_1 - 2x_2 + 3x_3 - 2y_1 + 4y_2 \leq 10 \\
& x \in \mathbb{Z}_+^3, \quad y \in \mathbb{R}_+^2
\end{align*}$$

3b. Write out explicitly the Benders reformulation; that is, the full master problem and the subproblem.