Particular Response, Multiple Input Terms

• Given the linear system

\[
\frac{d^n x}{dt^n} + a_n \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{d x}{dt} + a_0 x = f_1(t) + f_2(t) + f_3(t) + \cdots
\]

or

\[
\dot{x} = Ax + f_1(t) + f_2(t) + f_3(t) + \cdots
\]

• If \(x_k(t)\) is particular solution to system excited by \(f_k(t)\) alone, then particular solution to same system excited by \(f_1(t) + f_2(t) + f_3(t) + \cdots\) is \(x(t) = x_1(t) + x_2(t) + x_3(t) + \cdots\)

• Test: Substitute solution into equation

\[
\frac{d^n}{dt^n}\left[ x_1(t) + x_2(t) + x_3(t) + \cdots \right] + \\
+ a_n \frac{d^{n-1}}{dt^{n-1}}\left[ x_1(t) + x_2(t) + x_3(t) + \cdots \right] + \cdots + \\
+ a_2 \frac{d^2}{dt^2}\left[ x_1(t) + x_2(t) + x_3(t) + \cdots \right] + \\
+ a_1 \frac{d}{dt}\left[ x_1(t) + x_2(t) + x_3(t) + \cdots \right] + \cdots + a_0\left[ x_1(t) + x_2(t) + x_3(t) + \cdots \right] \\
= f_1(t) + f_2(t) + f_3(t) + \cdots
\]
• Rearrange, group terms:

\[
\left\{ \frac{d^r x}{dt^r} + a_{r-1} \frac{d^{r-1} x}{dt^{r-1}} + \cdots + a_1 \frac{d x}{dt} + a_0 x - f_1(t) \right\} + \\
\left\{ \frac{d^s x}{dt^s} + a_{s-1} \frac{d^{s-1} x}{dt^{s-1}} + \cdots + a_1 \frac{d x}{dt} + a_0 x - f_2(t) \right\} + \\
\left\{ \frac{d^t x}{dt^t} + a_{t-1} \frac{d^{t-1} x}{dt^{t-1}} + \cdots + a_1 \frac{d x}{dt} + a_0 x - f_3(t) \right\} + \cdots = 0
\]

• Differential equation within each curly bracket is zero, since \(x_k(t)\) is particular solution for \(f_k(t)\)

• Renders identity \(\{0\} + \{0\} + \{0\} + \cdots = 0\), devoid of derivatives!

• Solution: \(x(t) = x_1(t) + x_2(t) + x_3(t) + \cdots\)
Frequency Response Contains “Essence” of System Behavior

- Functions can be decomposed into sinusoidal components at various frequencies
- Basis of Fourier or spectral analysis
- **Example**: periodic excitation \( f(t) = f(t + T) \)

\[
f(t) = f(t+T) = f(t)
\]

- Repeats every period \( T \)
- From fourier series
  \[
f(t) = \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T} \right)
\]
  components @ frequencies: \( \omega_k = \frac{2\pi k}{T} \)

- Suppose:
  - \( x_k^{\cos}(t) \) solves for excitation \( \cos \omega_k t \)
  - \( x_k^{\sin}(t) \) solves for excitation \( \sin \omega_k t \)
- Solution for \( f(t) \)
  \[
x(t) = \sum_{k=1}^{\infty} \left[ a_k x_k^{\cos}(t) + b_k x_k^{\sin}(t) \right]
\]
Sinusoids: Good “test” functions

- Contained in complex exponentials $e^{st}$:
  
  $$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

  (note $e^{j\omega t} = e^{st} |_{s=j\omega}$) which solve linear differential equations with constant coefficients, excited by $e^{st}$ (see below)

- Periodic: repeats every T seconds

- Sines within limited range, for all time:
  
  $$-1 \leq \cos \omega t, \sin \omega t \leq 1$$

  ⇒ trace stays on oscilloscope screen

- Exponentials: approach 0 or $\infty$, for large t
Transfer Functions via $e^{st}$

- Linear differential eqns excited by $e^{st}$

\[ \dot{x} = Ax + Fe^{st} \]

\[ \frac{d^n x}{dt^n} + a_n \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{d x}{dt} + a_0 x = fe^{st} \]

- Solution: $x(t) = X(s)e^{st}$
- Substitute:

\[ \left( s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0 \right) X(s) e^s = f e^s \]

- Solve:

\[ G(s) = \frac{X(s)}{f(s)} = \frac{1}{s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0} \]

- Similar procedure for transfer functions from state equations
- Solution for exponential excitation $f(s)e^{st}$

\[ x(t) = X(s)e^{st} = G(s)f(s)e^{st} \]

- $f(s)$: amplitude of excitation
Frequency Response from Transfer Functions

- Given transfer function
  \[ G(s) = \frac{X(s)}{F(s)} \]
- Procedure, for frequency response from transfer functions:

1. Excite differential eqns. with
   - \( e^{j\omega t} = e^{st} \big|_{s=j\omega} \)
   - or: excite with \( e^{st} \), solve, then \( s = j\omega \)

2. Solution (previous result):
   \[ x(t) = [X(s)e^{st}] \big|_{s=j\omega} = [G(s)F e^{st}] \big|_{s=j\omega} \]
   \[ = G(j\omega)F e^{j\omega t} \]
   \[ = |G(j\omega)|e^{j \text{arg}[G(j\omega)]}F e^{j\omega t} \]
   \[ x(t) = F |G(j\omega)| e^{j\omega t + \text{arg}[G(j\omega)]} \]

3. For excitation by
   - \( \cos \omega t = \text{Re}(e^{j\omega t}) \) take Re:
     \[ x(t) = F |G(j\omega)| \text{Re}[e^{j \omega t + \text{arg}[G(j\omega)]}] \]
     \[ = F |G(j\omega)| \cos \{ \omega t + \text{arg}[G(j\omega)] \} \]
   - \( \sin \omega t = \text{Im}(e^{j\omega t}) \) take Im:
     \[ x(t) = F |G(j\omega)| \sin \{ \omega t + \text{arg}[G(j\omega)] \} \]