Analogous Systems

Identical differential equations & bond graph for
- Series electrical elements
- Parallel mechanical elements

Each system has
- Power input, effort source $S_e: e = V, F, T$
- Potential energy (of position) storage
  - Capacitance $C:C = C, k, K$
  - Displacement $q = q, x, \theta$
- Kinetic energy (of motion) storage,
  - Inertance $I:I= L, M, J$
  - Momentum $p = \lambda, p, h$
- Power loss, resistance $R: R = R, b, B$
- Common (same) flow, all elements: $f = i, v, \omega$
Identical differential equations & bond graph for
  ➢ Parallel electrical elements
  ➢ Series mechanical elements

Each system has
  ➢ Power input, flow source $S_f: f = i, v$
  ➢ Capacitance $C: C = C, k$
  ➢ Inertance $I: I = L, M$
  ➢ Power loss, resistance $R: R = R, B$
  ➢ Common (same) effort, all elements: $e = V, F$
Dynamic Systems Elements

Sources: supply power, prescribe effort or flow

Resistance: direct relation between effort & flow

Energy Storage Devices (single or multiports)
  ➢ Inertance: kinetic energy
  ➢ Capacitance: potential energy
  ➢ IC: kinetic & potential energies

Lossless (conserve power, \( \sum_{bonds} P_j = 0 \)) multiports
  ➢ Junctions
    o 0 junction: common effort, balance flows
    o 1 junction: common flow, balance efforts
  ➢ Transformers & Gyrators

For Bond graphs
  ➢ **Bond**: indicates power transfer between elements
  ➢ **Half arrow**: indicates direction of positive power transfer between elements
Sources

Effort source

\[ S_e : e \rightarrow e(t) \rightarrow A \]

- Prescribes effort \( e = e(t) \), effort labeled on half arrow side of bond
- Flow can be anything (rest of system determines flow)
- \( S_e \) prescribes effort (onto \( A \)), causal stroke (short vertical bar) away (ram against \( A \))

Flow source

\[ S_f : f \rightarrow f(t) \rightarrow A \]

- Prescribes flow \( f = f(t) \), non-half arrow side
- Effort can be anything
- \( S_f \) prescribes flow (to \( A \)), causal stroke toward (hose squirts away)
3.3. *BOND GRAPH METHODS*

An effort source *prescribes* its effort \( e(t) \) onto adjoining elements; the dependence \( e(t) \) on time \( t \) implies *prescription* of effort. Elements adjacent to the source must accept this imposed effort, but in return, these elements determine the flow back to the source. Figure 3.6a contains a bond graph of a generalized effort source \( S_e : e(t) \) imparting \( e(t) \) onto another element \( A \). The half arrow indicates positive power flow from \( S_e : e(t) \) to \( A \). Recall that in this text, we will always label efforts on the half arrow sides of bonds. Figures 3.6b to 3.6e depicts specific physical manifestations of effort sources for the various power domains mentioned in table 3.1. Each manifestation can be converted into bond graph form by replacing \( e(t) \) in figure 3.6a with the appropriate effort for that power domain, see table 3.2. For example, figure 3.6b shows the circuit schematic for a voltage source \( V(t) \); to obtain the bond graph form of figure 3.3b, we simply replaced \( e(t) \) in figure 3.6a with \( V(t) \). Physical devices that could be modeled as effort sources include a battery or AC electrical wall socket (voltage), a linear motor (force), a rotating motor or engine (torque), and a pump (pressure).

An effort source prescribes effort, but its flow depends on the reaction of other elements. For this reason, in figure 3.6a only the effort has been labeled on the bond emanating from \( S_e : e(t) \), since the flow is a priori unknown. In addition, introduced in figure 3.6a is the causal stroke—the short bar perpendicular to the bond, near the element “\( A \)”. The causal stroke in figure 3.6a asserts that the effort source \( S_e \) applies effort \( e(t) \) onto element \( A \), but its flow comes from or is determined by \( A \). On an effort source, the causal stroke is *always* positioned away from the effort.

---

**Figure 3.6:** a) A bond graph of an effort source \( S_e : e(t) \), with manifestations in the various energy domains: b) electrical—voltage source \( V(t) \); c) mechanical translational—applied force \( F(t) \); d) mechanical rotational—applied torque \( T(t) \); and e) fluidic—applied pressure \( P(t) \).
source $S_e : e(t)$; only this configuration is consistent with an effort source’s prescription of effort $e(t)$ onto adjoining elements. As a memory aid, the reader should imagine the causal stroke as a combination battering RAM and fire HOSE. The battering RAM, on the causal stroke side of a bond, batters the element it abuts with the bond’s effort; the fire HOSE, on the side opposite the causal stroke, squirts the element it faces with the flow pertinent to that bond. In figure 3.6a, the RAM abuts element A, indicating that effort source $S_e$ batters A with effort $e(t)$. The HOSE that points from A to $S_e$ suggests that A sends flow to $S_e$. Causality and its ramifications will be formally treated in section 3.3.4.

Flow and Flow Sources

![Flow Sources Diagram]

Figure 3.7: a) A bond graph of a flow source $S_f : f(t)$, with manifestations in energy domains: b) electrical—current source $I(t)$; c) mechanical translational—prescribed linear velocity $v(t)$; d) mechanical rotational—prescribed angular velocity $\Omega(t)$; and e) hydraulic—prescribed volumetric flow $Q(t)$.

Flow describes the movement or motion of a system. Flow $f$ manifests in the various power domains as: current $i$ (see Appendix section ??) in the electrical power domain, velocity $v$ in the mechanical translational power domain, angular velocity $\Omega$ in the mechanical rotational power domain, volumetric flow $Q$ in the fluidic power domain, and magnetic flux rate $\dot{\phi}$ in the magnetic power domain. The second row in table 3.1 lists the flows in the various power domains. Physical devices that could be modeled as flow sources include: current source (current), a massive translating inertia (velocity), a massive rotating flywheel or motor (angular velocity), and the blower on a hair dryer or an air conditioning system (volumetric flow). A flow source *prescribes* its flow
Resistance

➢ Direct relation between effort & flow:
   \( e = e(f) \) or \( f = f(e) \)

➢ If \( e \) vs. \( f \) plots in quadrants 1 & 3, dissipates power
   \( P = ef \geq 0 \)

➢ 2 Causality choices:

\[
\begin{matrix}
A & e \\
\rightarrow & \leftarrow \\
R & f = f(e)
\end{matrix}
\]

Effort Controlled:

\[ i = \frac{V}{R} \text{ form of Ohm’s law} \]

Action: A “rams” R with effort \( e \)
Reaction: R accepts effort \( e \) from A, then “hoses” A with flow \( f = f(e) \)
Flow Controlled:

\[ V = i R \quad \text{form of Ohm’s law} \]

Action: \( A \) “hoses” \( R \) with flow \( f \)
Reaction: \( R \) accepts flow \( f \) from \( R \), then \( R \) “rams” \( A \) with effort \( e = e(f) \)

- If \( e \) vs. \( f \) plots in quadrants 1 & 3, dissipates power \( P = e f \geq 0 \).
3.3. BOND GRAPH METHODS

Figure 3.10: Bond graphs of a resistance in its causal forms a) and b), with manifestations in energy domains: c) electrical—resistor; d) mechanical translational—linear dashpot; e) mechanical rotational—rotary dashpot; and f) hydraulic—flow constriction or turbulence.

Resistance

Reference [6], a text on electrical networks, states that an “element is called a resistor if the current through it, $i(t)$, and the voltage across it, $v(t)$, are related through an algebraic relation $g(v(t), i(t)) = 0$.” This definition includes resistors governed by Ohm’s law, diodes, and other devices. We will extend this definition to define a generalized resistance as a device wherein the flow $f_R$ and the effort $e_R$, are related through an algebraic relation

$$g(e_R, f_R) = 0. \quad (3.45)$$

When the relation between $e_R$ and $f_R$ given in equation 3.45 plots in the first or third quadrants—true for most real physical resistances—the power flow into the resistance $P_R = e_R f_R \geq 0$ is one-way, and the resistance dissipates power, converting energy into heat. Figure 3.10 depicts resistances in the various power domains: figure 3.10c shows an electrical resistor, which dissipates electrical power $V_R i_R$; figure 3.10d shows a linear dashpot, which dissipates mechanical translational power $F_b v_b$; figure 3.10e shows a rotary dashpot, which dissipates mechanical rotational power $T_b \Omega_b$; and figure 3.10f shows a flow constriction, which dissipates fluidic power $(P_1 - P_0) Q_c$ across the constriction. When a resistance dissipates power, the half arrow must always point towards the resistance, since system power (except heat) must always flow into the
Capacitance

- Kinematic constraint, flow $f_C$ & displacement $q$:
  $$f_C = \dot{q}$$

- Stores "potential" energy, energy of position or configuration

\[
U(q) = \int P \, dt = \int \dot{f} e \, dt = \int e \frac{dq}{dt} \, dt = \int e(q) \, dq
\]

=> Effort-displacement dependence $e = e(q)$

Linear capacitance: $e(q) = q/C$

\[
U(q) = \frac{q^2}{2C}
\]

- Energy variable: “displacement” $q$

- Note: $e = e(q) = \frac{\partial U}{\partial q}$

- Relates effort & displacement

\[\begin{align*}
\dot{f} &= \dot{q} \\
e &= e(q)
\end{align*}\]
## Capacitances in various power domains

<table>
<thead>
<tr>
<th>system</th>
<th>kinematics</th>
<th>displacement</th>
<th>effort</th>
<th>physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>general</td>
<td>( f = q )</td>
<td>( \dot q )</td>
<td>( e = e(q) )</td>
<td>Gauss law</td>
</tr>
<tr>
<td>electrical</td>
<td>( i = q )</td>
<td>charge ( q )</td>
<td>voltage ( V = V(q) )</td>
<td>C: capacitance</td>
</tr>
<tr>
<td>mech. translation</td>
<td>( v = x )</td>
<td>displacement ( x )</td>
<td>force ( F = F(x) )</td>
<td>k: elastic stiffness</td>
</tr>
<tr>
<td>mech. rotation</td>
<td>( \omega = \theta )</td>
<td>angular displacement ( \theta )</td>
<td>torque ( T = T(\theta) )</td>
<td>K: torsion stiffness</td>
</tr>
<tr>
<td>magnetic</td>
<td>( \phi = \dot \phi )</td>
<td>Magnetic flux ( \phi )</td>
<td>Magnetomotive force ( M = M(\phi) )</td>
<td>Magnetic ( R ): reluctance</td>
</tr>
<tr>
<td>fluidic</td>
<td>( Q = v )</td>
<td>fluid volume ( v )</td>
<td>pressure ( P = P(v) )</td>
<td>pressure ( P_t = v_t/(A/\rho g) )</td>
</tr>
</tbody>
</table>
rotational inertia; and for a fluidic system, $I$, the fluid inertance. There is no magnetic equivalent of inertance.

For inertances, flow and momenta dependencies can be linear or nonlinear. Examples of nonlinear inertances include an inductor with an iron core that saturates under large currents, limiting the flux linkage produced; and from the theory of relativity [5], the momentum $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$ of a mass $m$ depends non-linearly as velocity approaches the speed of light $c$. Bond graphs handle nonlinear elements as readily as linear elements.

Capacitance

![Capacitance Bond Graphs](image)

Figure 3.9: Bond graph of capacitances in a) integral causality (preferred), b) derivative causality with manifestations in energy domains: c) electrical—capacitor; d) mechanical translational—linear stiffness; e) mechanical rotational—torsional stiffness; and f) hydraulic—fluid tank.

A capacitance is a device that stores potential energy $\mathcal{E}_P$. To calculate the potential energy, we equate $\mathcal{E}_P$ to the total work $\int e \cdot dq$, performed by the effort $e$ applied through the displacement $q$, see the paragraph before equation 3.1.

Always associated with a capacitance is a kinematic constraint between the flow $f_C$ to the capacitor and the displacement $q$, via

$$f_C = dq/dt = \dot{q}. \quad (3.39)$$

Capacitances manifest in the various power domains as: electrical power domain, capacitor (figure 3.9c) where $i_c = \dot{q}$; mechanical translational power domain, stiffness (figure 3.9d) where $v_k =$
Multiport Capacitance

- Energy stored in field
- \( m \)-ports into Capacitance, \( m \)-power flows
  - Flows & displacement via kinematics:
    \[
    f_k = \dot{q}_k
    \]
  - Displacements: \( q_k \)
  - Power: \( P = \sum_{k=1}^{m} P_k = \sum_{k=1}^{m} e_k f_k \)
  - Total potential energy:
    \[
    E = \int P \, dt = \int \sum_{k=1}^{m} e_k f_k \, dt = \int \sum_{k=1}^{m} e_k \dot{q}_k \, dt = \int \sum_{k=1}^{m} e_k \, dq_k
    \]

\( \Rightarrow \) via integral, \( E = E(q_1, q_2, \ldots, q_m) \)
depends on all displacements $q_k$

- Energy & Power: $\frac{dE}{dt} = P = \sum_{k=1}^{m} e_k f_k$

- Derivative of $E = E(q_1, q_2, ..., q_m)$, chain rule:
  $$\frac{dE}{dt} = \sum_{k=1}^{m} \frac{\partial E}{\partial q_k} \frac{dq_k}{dt} = \sum_{k=1}^{m} \frac{\partial E}{\partial q_k} f_k$$

- Equate coefficients of $f_k$, in blue terms:
  $$e_k = e_k(q_1, q_2, ..., q_m) = \frac{\partial E}{\partial q_k}$$

- Effort on $k^{th}$ bond from partial of energy w.r.t. displacement $q_k$ on $k^{th}$ bond.
**Inertance**

- Physics constraint, effort $e_I$ & momentum $p$:
  $$ e_I = \dot{p} \quad \text{“inertial” force} $$
- Stores "kinetic" energy
  $$ T(p) = \int P \, dt = \int f \, e \, dt = \int f \, \frac{dp}{dt} \, dt = \int f(p) \, dp $$

  => Flow-momentum dependence  \hspace{1cm} f = f(p) \\
  
  Linear inertance:  \hspace{1cm} f(p) = p/M  \\
  \hspace{1cm} T(p) = p^2/2I $$

- Energy variable:  “momentum” $p$
- Note:  \hspace{1cm} f = f(p) = \partial T/\partial p$
- Relates flow & momentum

\[
\begin{align*}
  e &= \dot{p} \\
  f &= f(p) \\
  I \\
\end{align*}
\]
$f(t)$ onto adjoining elements; the effort $e$ to the flow source, determined by the other elements, can be anything. Again, the notation $f(t)$ with dependence on time $t$ implies prescription. Other elements bonded to the source must accept this imposed flow, but in return, determine the effort back to the source.

Figure 3.7a contains a bond graph of a flow source $S_f : f(t)$ imparting its general flow $f(t)$ onto another element $A$; the rest of figure 3.7 depicts manifestations of flow sources for other power domains mentioned in the second row of table 3.1. Analogous to the effort sources in section 3.3.3, each manifestation can be converted into bond graph form by replacing $f(t)$ in figure 3.7a with the appropriate flow for that power domain. The third row of table 3.2 lists these forms. For example, to convert the current source of figure 3.7b to bond graph form, we replace $f(t)$ in figure 3.7a with $I(t)$. Regarding causality, since a flow source $S_f$ prescribes flow, a flow source must have a HOSE pointing away from itself. With this causality, the flow source $S_f : f(t)$ in figure 3.7a applies flow $f(t)$ to adjoining element $A$. Element $A$ responds to the imposed flow via an effort applied back onto $S_f$. The battering RAM side of the causal stroke, which abuts $S_f$, implies this.

**Inertance**

![Bond graph of an inertance](image)

Figure 3.8: Bond graphs of an inertance in a) integral causality, and b) derivative causality, with manifestations in energy domains: c) electrical—inductance; d) mechanical translational—mass inertia; e) mechanical rotational—rotational inertia; and f) hydraulic—flow inertia. There is no magnetic or thermal inertia.

An inertance exhibits “inertia” behavior, wherein the device generates an inertial effort $e_I$ that
Table 3.3: Inertances for the various power domains used in this book. Magnetic systems, which lack inertial effects, were omitted.

<table>
<thead>
<tr>
<th>dynamics</th>
<th>general</th>
<th>electrical</th>
<th>mechanical translation</th>
<th>mechanical rotation</th>
<th>fluidic</th>
</tr>
</thead>
<tbody>
<tr>
<td>momentum</td>
<td>$e_I = \dot{p}$</td>
<td>$V_L = \lambda$</td>
<td>$F_I = \dot{p}$</td>
<td>$T_I = h$</td>
<td>$P = \dot{p}$</td>
</tr>
<tr>
<td>flow (linear I)</td>
<td>$f = f(p)$</td>
<td>$i = i(\lambda)$</td>
<td>$v = v(p)$</td>
<td>$\Omega = \Omega(h)$</td>
<td>$Q = Q(p)$</td>
</tr>
</tbody>
</table>

When more direct methods for extracting the constitutive law of equation 3.35, such as a plot of $f_I$ versus $p$, are not feasible, it is often convenient to use equation 3.37. This method involves calibrating the kinetic energy $E_K = E_K(p)$, and substituting into equation 3.37.

Equation 3.35 implies that a relationship exists between flow $f_I$ and momentum $p$. This relationship can be in the form of equation 3.35, or in an inverse form

$$ p = p(f_I) \quad (3.38) $$

For an inertance, two causalities are possible: integral causality shown in figure 3.8a, and described by equation 3.35, and derivative causality shown in figure 3.8b and described by equation 3.38. For both causalities, the dynamics of equation 3.33 applies. When the causal stroke is against the $I$, as in figure 3.8a, the inertance is in a state of integral causality, and the flow $f_I = f_I(p)$ depends on the momentum $p$. Integral causality with a RAM against the $I$ implies that the inertance accepts the effort $e_I = \dot{p}$ from the bond graph, and constructs its momentum $p$ via integration of $e_I$ over time. Using its constitutive relationship $f_I = f_I(p)$, equation 3.35, the inertance responds with a flow $f_I$. The HOSE squirting away from the $I$ in figure 3.8a suggests this. The constitutive relationship $f_I = f_I(p)$ may arise from equation 3.37, when the kinetic energy is known. When the causal stroke is away from the $I$, as in figure 3.8b, the inertance is in a state of derivative or dependent causality. As the causal picture suggests, the HOSE squirting flow $f_I$ into $I$ necessitates a momentum dependence $p = p(f_I)$ on flow; consequently, effort $e_I = \dot{p}$, must depend on $f_I$. Important physics and system design information is often present whenever derivative causality appears in a bond graph.

Inertia like behavior manifests in the electrical power domain as inductance, in the mechanical translational power domain as mass, in the mechanical rotational power domain as rotational inertia, and in the fluidic power domain as fluid inertance. Table 3.3 summarizes, and figures 3.8c-f depicts these forms. Inertance constants for linear inertial elements in the different power domains, presented in table 3.2, are: for an electrical system, $L$, the electrical inductance; for a mechanical translational system, $m$, the mass inertia; for a mechanical rotational system, $J$, the
# Inertances in various power domains

<table>
<thead>
<tr>
<th>system type</th>
<th>physics</th>
<th>momentum</th>
<th>flow dependence</th>
<th>Physics law</th>
</tr>
</thead>
<tbody>
<tr>
<td>general</td>
<td>( e = \dot{p} )</td>
<td>( p )</td>
<td>( f = f(p) )</td>
<td>Faraday</td>
</tr>
<tr>
<td>electrical</td>
<td>Inductor voltage ( V = \dot{\lambda} )</td>
<td>flux linkage ( \lambda )</td>
<td>current ( i = i(\lambda) )</td>
<td>Newton</td>
</tr>
<tr>
<td>mech. translation</td>
<td>inertial force ( F_I = \dot{p} )</td>
<td>linear mom. ( p )</td>
<td>velocity ( v )</td>
<td>( F = ma = \dot{p} )</td>
</tr>
<tr>
<td>mech. rotation</td>
<td>inertial torque ( T_I = \dot{h} )</td>
<td>ang. mom. ( h )</td>
<td>ang.vel. ( \omega )</td>
<td>( T = I\alpha = \dot{h} )</td>
</tr>
<tr>
<td>fluidic</td>
<td>inertial pressure ( P_I = \dot{p} )</td>
<td>fluidic momentum ( p )</td>
<td>fluid volume ( Q = Q(p) )</td>
<td>unsteady flow terms in momentum equations</td>
</tr>
</tbody>
</table>
Multiport Inertance

\[ e_1 \quad f_1 \quad I \]
\[ f_k \]
\[ e_k \quad f_k \quad e_m \quad f_m \]

\[ \Rightarrow \text{ via integral, } \quad E = E(p_1, p_2, ..., p_m) \]

\[ f_k = f_k(p_1, p_2, ..., p_m) = \frac{\partial E}{\partial p_k} \]

- Effort on \( k^{th} \) bond from partial of energy w.r.t. displacement \( q_k \) on \( k^{th} \) bond.
IC Device

- Stores kinetic & potential energies in same “field”

\[ E = E(p_1, p_2, ..., p_m, q_1, q_2, ..., q_n) \]

- Ports with momenta \( p_k \) & displacements \( q_l \)
- Flows on I bonds:
  \[ f_k = \frac{\partial E}{\partial p_k} = f_k(p_1, p_2, ..., p_m, q_1, q_2, ..., q_n) \]
- Efforts on C bonds:
  \[ e_l = \frac{\partial E}{\partial q_l} = e_l(p_1, p_2, ..., p_m, q_1, q_2, ..., q_n) \]
0 & 1 Junctions

- No power loss or storage ⇒ Power balance:

\[
P = \sum_{k=1}^{n} P_{k}^{in} - \sum_{i=1}^{m} P_{i}^{out} = \sum_{k=1}^{n} e_{k}^{in} f_{k}^{in} - \sum_{i=1}^{m} e_{i}^{out} f_{i}^{out} = 0
\]

0 junction: common (same) effort, all bonds:

\[
e_{1} = e_{2} = \ldots = e_{n} = e_{m} = e
\]

\[
\sum_{k=1}^{n} f_{k}^{in} - \sum_{i=1}^{m} f_{i}^{out} = \sum_{k=1}^{n+m} f_{k} = 0
\]

⇒ Flow balance

0 junction incorporates:
- Electrical Kirchoff’s Current Law
  ( \sum \text{currents into} = 0 )
  node
- Mechanical kinematics (balance of velocities & rate of displacements)
CHAPTER 3. MODELING PHYSICAL SYSTEMS

Equilibrium of forces, \( \sum_{k=1}^{n} F_k = 0 \), for mechanical translational domains, wherein the sum of forces \( F_k \) on a body along some direction must equal zero.

Equilibrium of moments, \( \sum_{k=1}^{n} T_k = 0 \), for mechanical rotational domains, wherein the sum of moments \( T_k \) over a body along some axis must equal zero.

Momentum equation, \( \sum_{k=1}^{n} P_k = 0 \), for fluidic power domains, wherein the sum of the pressure drops \( P_k \) along a flow path must equal zero.

Magnetomotive force equilibrium, \( \sum_{k=1}^{n} M_k = 0 \), for magnetic power domains, wherein the sum of the magnetomotive force drops \( M_k \) along a flux path must equal zero.

Note that inertial effects such as \( F_I = \dot{p} \) arise as separate terms in these balances.

In like manner—see table 3.2—the flow balancing property of a 0 junction, equation 3.51, programs into bond graphs the following:

Kirchoff’s current law, \( \sum_{k=1}^{n} i_k = 0 \), for electrical power domains, wherein the sum of the currents \( i_k \) flowing into a circuit node must equal zero.

Translation kinematics \( \sum_{k=1}^{n} v_k = 0 \), for mechanical translational domains, which equates translational velocities \( v_k \) along some direction across a body to zero.

Rotational kinematics \( \sum_{k=1}^{n} \Omega_k = 0 \), for mechanical rotational domains, wherein the rotational velocities \( \Omega_k \) along some axis through a body must equate to zero.

Continuity equation \( \sum_{k=1}^{n} Q_k = 0 \), for incompressible fluidic power domains, wherein the sum of the volumetric flows \( Q_k \) into and out of a control volume must equate to zero.

Flux rate continuity equation \( \sum_{k=1}^{n} \dot{\phi}_k = 0 \), for magnetic power domains, wherein the sum of the flux flows \( \dot{\phi}_k \) over a node in a magnetic circuit must equal zero.

Note that capacitance effects such as fluid storage in control volumes will appear as separate terms in the flow balance. As we shall see in chapter ??, state equations can be extracted from the bond graph using the equality or balance properties (see equations 3.50 and 3.51) of 0 and 1 junctions. This extraction is made easier via causality information, studied next.

3.3.4 Causality: Cause and Effect

Present on any bond connecting two elements are the power conjugate variables, effort \( e \) and flow \( f \). Two variables are power conjugate to each other whenever their product equals the power flowing over that bond. Through these power conjugate variables, elements interact. With two
1 junction: common (same) flow, all bonds:
\[ f_1 = f_2 = \ldots = f_n = f_m = f \]
\[ \sum_{k=1}^{n} e_{in}^k - \sum_{i=1}^{m} e_{out}^i = \sum_{k=1}^{n+m} e_k = 0 \]

1 junction incorporates:
- Electrical Kirchoff's Voltage Law (over loop)
- Mechanical D'Alembert's dynamic equilibrium
Equilibrium of forces, $\sum_{k=1}^{n} F_k = 0$, for mechanical translational domains, wherein the sum of forces $F_k$ on a body along some direction must equal zero.

Equilibrium of moments, $\sum_{k=1}^{n} T_k = 0$, for mechanical rotational domains, wherein the sum of moments $T_k$ over a body along some axis must equal zero.

Momentum equation, $\sum_{k=1}^{n} P_k = 0$, for fluidic power domains, wherein the sum of the pressure drops $P_k$ along a flow path must equal zero.

Magnetomotive force equilibrium, $\sum_{k=1}^{n} M_k = 0$, for magnetic power domains, wherein the sum of the magnetomotive force drops $M_k$ along a flux path must equal zero.

Note that inertial effects such as $F_I = \dot{p}$ arise as separate terms in these balances.

In like manner—see table 3.2—the flow balancing property of a 0 junction, equation 3.51, programs into bond graphs the following:

Kirchoff’s current law, $\sum_{k=1}^{n} i_k = 0$, for electrical power domains, wherein the sum of the currents $i_k$ flowing into a circuit node must equal zero.

Translation kinematics $\sum_{k=1}^{n} v_k = 0$, for mechanical translational domains, which equates translational velocities $v_k$ along some direction across a body to zero.

Rotational kinematics $\sum_{k=1}^{n} \Omega_k = 0$, for mechanical rotational domains, wherein the rotational velocities $\Omega_k$ along some axis through a body must equate to zero.

Continuity equation $\sum_{k=1}^{n} Q_k = 0$, for incompressible fluidic power domains, wherein the sum of the volumetric flows $Q_k$ into and out of a control volume must equate to zero.

Flux rate continuity equation $\sum_{k=1}^{n} \dot{\phi}_k = 0$, for magnetic power domains, wherein the sum of the flux flows $\dot{\phi}_k$ over a node in a magnetic circuit must equal zero.

Note that capacitance effects such as fluid storage in control volumes will appear as separate terms in the flow balance. As we shall see in chapter ??, state equations can be extracted from the bond graph using the equality or balance properties (see equations 3.50 and 3.51) of 0 and 1 junctions. This extraction is made easier via causality information, studied next.

### 3.3.4 Causality: Cause and Effect

Present on any bond connecting two elements are the power conjugate variables, effort $e$ and flow $f$. Two variables are power conjugate to each other whenever their product equals the power flowing over that bond. Through these power conjugate variables, elements interact. With two
Causality: 0 & 1 Junctions

➢ 0 junction common effort
   \[ e_1 = e_2 = \ldots = e_n = e \]
   \[ \Rightarrow \text{only ONE bond can set common effort } e \]

\[ \Rightarrow \text{ONE ram against 0 (otherwise contradiction)} \]

Note: 0 junction has only 1 ram, but **MUST** have

➢ 1 junction: common flow
   \[ f_1 = f_2 = \ldots = f_n = f \]
   \[ \Rightarrow \text{only ONE bond can set common flow } f \]

\[ \Rightarrow \text{ONE hose squirts 1 (otherwise contradiction)} \]

Note: 1 junction has only 1 hose, but **MUST** have
Transformers & Gyrators

➤ Converts power, spans domains
➤ Lossless 2 port: \( P_1 = P_2 \)

➤ Transformer

\[
\begin{align*}
e_1 & \quad \text{TF: } n \quad e_2 \\
f_1 & \quad f_2
\end{align*}
\]

○ relates effort to effort: \( e_1 = n e_2 \)
○ & flow to flow: \( f_2 = n f_1 \)

○ conserves power:
\[
(ne_2) f_1 = e_1 f_1 = P_1 = P_2 = e_2 f_2 = e_2 (n f_1)
\]

➤ Gyrator

\[
\begin{align*}
e_1 & \quad \text{GY: } r \quad e_2 \\
f_1 & \quad f_2
\end{align*}
\]

○ relates effort to flow: \( e_1 = r f_2 \) & \( e_2 = r f_1 \)

○ conserves power:
\[
(rf_2) f_1 = e_1 f_1 = P_1 = P_2 = e_2 f_2 = (r f_1) f_2
\]
3.3. BOND GRAPH METHODS

Figure 3.11: Bond graphs of transformers in its allowed causal forms a) and b), with manifestations in energy domains: c) electrical—transformer with turns ratio \( n = n_1/n_2 \); d) mechanical translational—lever mechanism with leverage \( n = \ell_2/\ell_1 \); e) mechanical rotational—gears and rollers with gear ratio \( n = R_1/R_2 \). Transformers can also span power domains. Examples include f) translational to hydraulic—piston with \( n = A \); g) rotational to translational—roller on flat, or rack and pinion with \( n = 1/R \); and h) rotational to hydraulic—positive displacement pump.

in an electrical transformer. A bond graph transformer relates the efforts on all bonds connected to it. If the transformer has two ports, the efforts \( e_1 \) and \( e_2 \) on the bonds pertaining to those ports are related through the transformer modulus \( n \) according to the first of the following equations:

\[
e_1 = ne_2, \quad f_2 = nf_1. \tag{3.46}
\]

If no power is lost or stored as energy in the transformer—*if the transformer is ideal*—the power \( P_1 = e_1f_1 \) flowing into the transformer at port 1 must equal the power \( P_2 = e_2f_2 \) flowing out of the transformer at port 2. Equation 3.2 gives

\[
e_1f_1 = e_2f_2. \tag{3.47}
\]

As a consequence of equation 3.47 and the first of equations 3.46, flows \( f_1 \) and \( f_2 \) must be related. If we substitute the first of equations 3.46 into equation 3.47, the second of equations 3.46 results,
Figure 3.12: Bond graphs of gyrators in allowed causal forms a) and b), with examples: c) electrical: electrical gyrator formed by matched pairs of field effect transistors, or transconductance amplifiers; d) electrical-mechanical rotational: DC servo motor; e) mechanical translational-mechanical rotational: gyroscope; and f) electrical-magnetic: solenoid. Gyrators often span power domains.

The two allowable causal forms for a 2-port gyrator are shown in figures 3.12a and 3.12b. These arrangements, and only these arrangements, are consistent with the effort to flow and flow to effort relationships (see equations 3.48) germane to a gyrator. The RAMs and HOSEs formed by the placement of causal strokes in figure 3.12a suggests that effort $e_1$ induces flow $f_2$ (from left to right, RAM in gives HOSE out), or effort $e_2$ induces flow $f_1$ (from right to left, RAM in gives HOSE out). In figure 3.12b, flow $f_1$ induces effort $e_2$ (from left to right, HOSE in gives RAM out), or flow $f_2$ invokes effort $e_1$ (from right to left, HOSE in gives RAM out).

Like transformers, gyrators can be further generalized. Gyrators can have multiple terminals, linking multiple efforts to multiple flows. These efforts and flows are often arranged into a vector of efforts and a vector of flows. If all responses of the gyrator are linear, a matrix of moduli $R$ replaces the scalar modulus $r$. Also, a gyrator can be modulated, wherein the modulus $r = r(\alpha)$ is controlled by another parameter or variable $\alpha$. 

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TF & GY Causality

Transformer: \( e_1 = n e_2 \) effort evokes effort

\[ f_1 = n^{-1} \quad f_2 \quad \text{flow evokes flow} \]

\( \Rightarrow 2 \) choices:

\[
\begin{array}{c}
\begin{array}{c}
\text{Transformer} \\
\hline
\begin{array}{c}
e_1 \quad \text{TF} \quad e_2 \\
\hline
f_1 \quad f_2
\end{array}
\end{array}
\end{array}
\]

Gyrator: \( e_1 = r f_2 \) flow evokes effort

\[ f_1 = r^{-1} \quad e_2 \quad \text{effort evokes flow} \]

\( \Rightarrow 2 \) choices

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
ge_1 \quad \text{GY} \quad e_2 \\
\hline
f_1 \quad f_2
\end{array}
\end{array}
\end{array}
\]
### Table 3.2: A list of bond graph elements for various energy domains, with corresponding element constants and SI units in square brackets.

<table>
<thead>
<tr>
<th>Element</th>
<th>General</th>
<th>Electrical</th>
<th>Mechanical Translation</th>
<th>Mechanical Rotation</th>
<th>Fluidic</th>
<th>Magnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort Source</td>
<td>$S_e : e(t)$</td>
<td>$S_e : V(t)$</td>
<td>$S_e : F(t)$</td>
<td>$S_e : T(t)$</td>
<td>$S_e : P(t)$</td>
<td>$S_e : J$</td>
</tr>
<tr>
<td>Flow Source</td>
<td>$S_f : f(t)$</td>
<td>$S_f : I(t)$</td>
<td>$S_f : v(t)$</td>
<td>$S_f : \Omega(t)$</td>
<td>$S_f : Q(t)$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>Capacitance</td>
<td>$C$</td>
<td>$C$ [F]</td>
<td>$k$ [N m$^{-1}$]</td>
<td>$\kappa$ [N m rad$^{-1}$]</td>
<td>$C_t$ [m$^4$ N$^{-1}$]</td>
<td>$R$ [V A$^{-1}$]</td>
</tr>
<tr>
<td>Inertance</td>
<td>$I$</td>
<td>$L$ [H]</td>
<td>$m$ [kg]</td>
<td>$J$ [kg m$^2$]</td>
<td>$\mathcal{I}$ [N m$^{-2}$ s]</td>
<td>$-$</td>
</tr>
<tr>
<td>Resistance</td>
<td>$R$</td>
<td>$R$ [Ω = V A$^{-1}$]</td>
<td>$b$ [N m$^{-1}$ s]</td>
<td>$\beta$ [N m s]</td>
<td>$R_c$ [N m$^{-2}$ s]</td>
<td>$R_m$ [Ω$^{-1}$]</td>
</tr>
<tr>
<td>Transformer</td>
<td>$TF$</td>
<td>transformer</td>
<td>levers</td>
<td>gears &amp; rollers</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>Gyrorator</td>
<td>$GY$</td>
<td>transconductance amplifiers</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>0 Junction</td>
<td>$\sum_k f_k = 0$</td>
<td>Kirchhoff’s Current Law</td>
<td>kinematics: velocities</td>
<td>kinematics: angular velocities</td>
<td>Continuity equation</td>
<td>flux conservation</td>
</tr>
<tr>
<td>1 Junction</td>
<td>$\sum_k e_k = e_{in}$</td>
<td>Kirchhoff’s Voltage Law</td>
<td>Equilibrium: forces</td>
<td>Equilibrium: moments</td>
<td>Momentum equation</td>
<td>magnetic force conservation</td>
</tr>
</tbody>
</table>

(easily done), the resulting equations will be wrong. Finally, we note that similar to section 3.3.1, the governing equations for the mechanical translational problem can be extracted from the bond graph. We will do this in the next chapter.

### 3.3.3 Dynamic System Elements

In this section, dynamic elements will be described and classified. Table 3.2 lists important bond graph elements for each type of energy domain studied in this book. For each element we will present its schematic, describe its physics and behavior, and then generalize the element with a bond graph representation. We will extend the bond graph generalization to the following power domains: electrical, mechanical translational, mechanical rotational, and hydraulic. For completeness, magnetic elements—covered in chapter ??—have been included in the table.

**Effort and Effort Sources**

Effort provides the impetus or drive needed to put a system into motion. Effort $e$ manifests in the various power domains as: voltage $V$ (see Appendix section ?? for a definition) in the electrical power domain, force $F$ in the mechanical translational power domain, torque $T$ in the mechanical rotational power domain, pressure $P$ in the fluidic power domain, and magnetomotive force $M$ in the magnetic power domain. The first row of table 3.1 lists the efforts in the various power domains, with SI units enclosed within square brackets.