Modeling and Experimentation: Two-Can System

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One and two-can system dynamics

Consider a can (or tank) filled to a known level, emptying into another can that is initially empty. From experience, we can guess that the dynamic response of each can volume will follow trends as shown below.

What does it take to build a system model that can accurately predict critical measures of performance?
Lab Objectives

- Extend skill in physical system modeling and experimentation
- Design experiments that help parameterize a system model
- Verify and validate models using measurements

For the two-can system:

1. Use a model to predict the initial volume of water in the top can that will result in the level of water in the bottom can reaching a specified level at some point in time

2. Determine whether you can ‘exactly’ maximize the level in the bottom can without spilling.

3. Use direct measures of level or volume over time to assess the overall model correctness, such as times to peak, shape of trends, etc.
Tank emptying problems in physics classes

**EXAMPLE 12**

A water tank has a (small) hole near its bottom at a depth of 2.0 m from the top surface (see Fig. 18.31). What is the speed of the stream of water emerging from the hole?

**SOLUTION:** Qualitatively, one of the streamlines for the water flowing out of the tank will look as shown in Fig. 18.31. Since the hole is small, the water level at the top of the tank drops only very slowly; we can therefore take \( v_y = 0 \) in Eq. (18.25). Furthermore, the pressures at the top and in the emerging stream of water are the same; both are equal to the atmospheric pressure \( p_0 \). Thus, \( p_1 = p_2 = p_0 \). With this, Eq. (18.25) becomes

\[
\frac{1}{2} \rho v_y^2 + \rho g y_2 + p_0 = \rho g y_1 + p_0
\]

(18.28)

We can cancel the terms \( p_0 \), and we can move the term \( \rho g y_2 \) to the right side of the equation:

\[
\frac{1}{2} \rho v_y^2 = \rho g y_1 - \rho g y_2
\]

(18.29)

If we divide both sides by \( \frac{1}{2} \rho \) and extract the square root of both sides, we find

\[
v_2 = \sqrt{2g(y_1 - y_2)}
\]

(18.30)

With \( y_1 - y_2 = 2.0 \text{ m} \), the speed is

\[
v_2 = \sqrt{2 \times 9.81 \text{ m/s}^2 \times 2.0 \text{ m}} = 6.3 \text{ m/s}
\]

**COMMENT:** According to Eq. (18.30), the speed of the emerging water is exactly what it would be if the water were to fall freely through a height \( y_1 - y_2 \) [compare Eq. (2.29)]. This result is called Torricelli’s theorem. It merely expresses conservation of energy: when a drop of water flows out at the bottom, the loss of potential energy of the water in the tank is equivalent to the removal of a drop of water from the top; the conversion of this potential energy into kinetic energy will give the drop the speed of free fall. We have already relied on this energy argument in Chapter 8, where we calculated the speed of water flowing out of a pipe in a hydroelectric storage plant.

*From Physics for Engineers and Scientists, Ohanian and Markert, 3rd ed*
From: Introduction to Fluid Mechanics, Fox and McDonald, 4th edition

4.48 Water enters the tank through a pipe at rate \( Q = 0.35 \text{ ft}^3/\text{sec} \). Water also leaves the tank through a smoothly rounded nozzle of 2 in. diameter. The exit velocity through this hole, which depends on the height, \( h \), of the water level above the hole, is \( V_{\text{exit}} = \sqrt{2gh} \). At \( t = 0 \), \( h = 9 \text{ ft} \). Derive an equation that could be solved for \( h(t) \) when \( h > 4 \text{ ft} \). For \( 4 < h < 5 \text{ ft} \), is \( \frac{dh}{dt} \) greater than, less than, or equal to zero?

In this example, control volume equations are used to derive an ODE in terms of the height, \( h \), of fluid in this tank.
Build a model for our purposes
By applying the *mass continuity equation* for a control volume on a can,

\[ \dot{m}_{\text{stored}} = \sum \dot{m}_{\text{in}} - \sum \dot{m}_{\text{out}} \]

and assuming incompressible flow,

\[ \rho \dot{V} = \sum \rho Q_{\text{in}} - \sum \rho Q_{\text{out}} \]

So for a one can system,

\[ \dot{V} = \sum Q_{\text{in}} - \sum Q_{\text{out}} \]

where \( V \), the volume stored in the can, is the system state.

\[ \rightarrow \] This equation cannot be solved until the terms on the right-hand side are known either as functions of states or as inputs.
Building *systems* model

When systems modeling, we define basic ‘elements’ that aid us in composing the model equations. For example, if using a bond graph approach, each can would be represented by:

- Potential energy stored by fluid contained in the can, modeled by a \( C \) element
  - quasi-static assumption
  - constant-area tank
  - pressure-volume \((P – V)\) constitutive relation
- Energy dissipation related to flow exiting orifice, modeled by a \( R \) element
  - Steady-flow Bernoulli model
  - simplifying assumptions
  - pressure-flow \((P – Q)\) constitutive relation
- Any other flows coming into the system.
A can only stores potential hydraulic energy (if we can ignore inertial effects), and we show that this is a capacitive element by recalling that,

\[ P = \rho gh = \frac{\rho g}{A}V = \frac{1}{C}V \]

where the can is assumed to have constant cross-sectional area, so \( C \) is a hydraulic capacitance, \( C = A/(\rho g) \).

If the can or tank does not have constant cross-sectional area, as is common in many applications as illustrated below, then the constitutive law would not be linear.

It may also be the case that you cannot ignore inertial effects of the fluid. What would the tank look like in such a case?
An Orifice/Valve as an $\mathbf{R}$ Element

You can show (see Appendix) that orifice flowrate, $Q$, can be directly related to pressure drop across the orifice, $P$, by,

$$Q = K_o A_o \text{sgn}(P) \sqrt{2|P|/\rho}$$

where $K_o$ is an ideal (theoretical) flow coefficient, $A_o$ the orifice area. The ‘sgn’ function allows this flow equation to be used even if $P$ becomes negative, but for the can system this is not necessary.

We identify the flowrate as a flow variable and pressure as an effort variable, the product being the dissipated power, $\mathcal{P} = P \cdot Q$. Since the orifice involves dissipative processes, we would model the orifice as a resistive ($\mathbf{R}$) element.

**Note:** Refer to the detailed discussion of the orifice flow rate model in the appendix, especially to see how you can calculate an ideal flow coefficient, $K_o$. 
To derive the state equation for the one-can system, use the constitutive relations discussed earlier. First,

\[
\frac{dV}{dt} = Q_{in} - Q_{out}
\]

where

\[
Q_{out} = K_o A_o \sqrt{2|P|/\rho}.
\]

But,

\[
P = V/C
\]

Now,

\[
\frac{dV}{dt} = Q_{in} - K_o A_o \sqrt{2|V|/\rho C}
\]

which can be simplified to a first-order state equation,

\[
\frac{dV}{dt} = Q_{in} - K \sqrt{V}, \quad V \geq 0
\]

where \( K \) is a constant for a given can.

The only parameter needed for a one can system is \( K \). Note that \( K \) is a composite parameter that integrates can and orifice geometry, fluid properties, and an *ideal* orifice flow coefficient. This is the only parameter we need for a given can.
How do you find $K$?

Consider two options for finding $K$:

- Calculate a theoretical or ideal $K$ as shown in the Appendix, where,

$$K_{\text{ideal}} = K_{o,\text{ideal}} A_o \sqrt{2g/A_{\text{can}}},$$

  with $A_o$ is orifice area, $A_{\text{can}}$ is can area, and,

$$K_{o,\text{ideal}} = 1/ \left[ 1 - (A_o/A_{\text{can}})^2 \right]$$

- Measure values of $K$ from experiments.

The first case is straightforward and an example is given in the Appendix. It should be kept in mind that these values would ignore influence of the non-ideal effects in the orifice flow. Let's consider *physical design of experiments* for finding $K$ experimentally.
Two cases: steady-state and dynamic emptying

A steady-state condition occurs when $Q_{in}$ is equal to $Q_{out}$ as illustrated below.

\[
\frac{dV}{dt} = Q_{in} - Q_{out} = 0.
\]

The system is said to be in equilibrium or steady-state, and volume of fluid in the can will remain constant.

The more general dynamic state occurs when $Q_{in} \neq Q_{out}$.

The tank empties and/or fills over time, with a model given by,

\[
\frac{dV}{dt} = Q_{in} - K \sqrt{V}
\]
Steady-state versus dynamic emptying

**Steady-state:** In general, setting \( \frac{dx}{dt} = 0 \) for all your state equations gives \( n \) algebraic equations that can be solved for the equilibrium states.

In this case, \( \frac{dV}{dt} = 0 \), giving,

\[
Q_{in} = Q_{out},
\]

which can be solved for \( V_e \), the steady-state or equilibrium volume.

Q. Can you solve for \( V_e \)?

**Dynamic state:** The special case \( Q_{in} = 0 \) with known \( V_o \) is purely a can emptying problem,

\[
\frac{dV}{dt} = -K \sqrt{V}
\]

which can be solved for \( V(t) \) in closed form, given we know \( K \).

Q. Can you find \( V(t) \)?
Finding $K$ from can emptying experiments

The emptying can model can be used to design an experiment to determine the unknown parameter, $K$.

The equation,

$$\frac{dV}{dt} = -K\sqrt{V}$$

is a nonlinear ordinary differential equation that can be integrated from the separable form,

$$\int \frac{dV}{\sqrt{V}} = -\int K \, dt$$

An ‘emptying experiment’ begins with an initial volume, $V_o$, that empties in time, $T_e$, as expressed through the integration,

$$\int_{V_o}^{0} \frac{dV}{\sqrt{V}} = -\int_{0}^{T_e} K \, dt$$
Model analysis

Integration of the equation,

\[ \int_{V_o}^{0} \frac{dV}{\sqrt{V}} = - \int_{0}^{T_e} K \, dt \]

yields,

\[ \int_{V_o}^{0} \frac{dV}{\sqrt{V}} = 2V^{1/2}|_{V_o}^{0} = - \int_{0}^{T_e} K \, dt = -Kt|_{0}^{T_e} \]

giving,

\[ 2\sqrt{V_o} = K T_e. \]

This relation suggests that measurements of initial volume and time-to-empty can be used to estimate the parameter \( K \).

This is as an example of a model-based experiment design. The model guides our selection of quantities to measure in the laboratory: a) initial volume, \( V_o \), and b) time to empty, \( T_e \). These quantities can be readily measured using simple instruments.

ME 144L Dynamic Systems and Controls Lab (Longoria)
The relation, $2\sqrt{V_o} = KT_e$, suggests there is a linear relation between initial volume and time-to-empty.

\[ \sqrt{V_o} = \frac{K}{2} \cdot T_e \]

Mirror-image the data to force data through the (0,0) point.

The model should force a (0,0) point - this is realistic!

Q. Do you think it is correct to force the trend line in this relation to go through (0,0)?
Determining $K$ from a steady-state experiment

The other way to find a $K$ value is to use a steady-state experiment.

1. Describe what measurements would be needed to find $K$.
2. Explain how the physical conditions are different from those in a dynamic can emptying experiment.
3. Is there a preference in choosing one of the experimental approaches described over the other?
Alternative measurement approaches and assessment

The one-can model and experiments motivate measuring initial volumes, $V_o$, and times-to-empty, $T_e$, to determine $K$ through a model-based correlation of $2\sqrt{V_o/T_e}$.

Direct sensing of fluid level or volume would enable:

1. assessing how well predicted values compare to measured values over time,
2. determining if critical values are accurately predicted (e.g., peak volumes, time-to-peaks, etc.), and
3. ways to find $K$ values during actual operation of a system (detecting changes, faults, etc.)
Example of a ‘brute force’ adjustment of $K$

Given measured volume-vs-time data, you can compare to results from a direct simulation of a 1-can system and adjust $K$ value to improve overall ‘fit’.

Aside from visual inspection only, a ‘goodness-of-fit’ to the model could be directly calculated. For example, calculate the sum of the squared errors for all points. Adjustment of $K$ can then be made by manually minimizing this sum.
There are many ways to measure or infer fluid level in a tank or can.

**Common level measurement sensors**

(a) float switch level detector

(b) resistive level detectors

(c) capacitive level sensor

Below is a resistive level sensor for the two-can system.

The resistance (or impedance) of the water between the probes varies with the inverse of fluid level. As height drops, resistance goes up.
Pressure can be used to infer volume in a can

Since,

\[ P = \rho gh = \frac{\rho g}{A} Ah = \frac{1}{C} V \]

where \( C = A/\rho g \).
Pressure sensors

Most pressure sensors feature a diaphragm that responds to applied pressure.

In the example to the right, the diaphragm contacts a small beam instrumented with strain gauges.

The strain-gauge sensing mechanism converts the mechanical response to an electrical signal proportional to pressure.

(Strain gauges will be studied in another lab.)
PX409 pressure sensor

Pressure sensor from Omega Engineering used in 144L lab

The diaphragm in the PX409 pressure sensor is micro-machined to include piezoresistive strain gauges. Strain gauges will be studied in a future lab.

The diagram below shows how MEMS pressure sensors combine the diaphragm/strain design.
Two-can setup with pressure sensing
Using off-the-shelf sensors

Some key issues when selecting and/or using off-the-shelf sensors, such as the pressure sensors to be used in this lab study include the following:

1. Be sure that the sensor range of operation meets the requirements. This refers to the input to the sensor, namely the quantity to be measured, but also to the output signals which must match to the system being used to make measurements. Be aware that sensors can have many different ways for conveying the measurand value(s) in the output signal (voltage, current, frequency, etc.).

2. Make sure to understand the power requirements of the sensor. If you are working on the bench, sometimes this is not an issue since you may have sufficient power from available electrical power supplies. This may not be the case for mobile platforms. Understand both the voltage and current requirements.

3. Make sure to make the right signal connections.
Two-can system model

The state equations for the two can system are found by simply applying continuity on the ‘second’ can.

You should be able to show that the state equations for the volumes in cans 1 (top) and 2 (bottom) are, respectively,

\[
\begin{align*}
\dot{V}_1 &= -K_1 \sqrt{V_1} \\
\dot{V}_2 &= +K_1 \sqrt{V_1} - K_2 \sqrt{V_2}
\end{align*}
\]

where the \( K \) values are distinct for each can. Note that there is no input flow to top can in this system.

Unlike for the ‘one-can’ model, these equations cannot be readily solved analytically. Numerical simulations will show that the expected trends can be predicted with this model, but accuracy is only achieved with accurate model parameters and initial conditions.
Two-can system block diagram

A block diagram for the two-can system is shown below.

One way to keep volume greater than or equal to zero is to use a saturation block with a lower limit set to zero and upper limit set to Inf.
Script for two-can

The two-can system ODEs can be formatted in script form as shown below:

```matlab
function f = two_can(t,V)
global K1 K2
if V(1)<0
    Q1 = 0; % if V<0, can is empty
else
    Q1 = K1*sqrt(V(1));
end
if V(2)<0
    Q2 = 0; % if V<0, can is empty
else
    Q2 = K2*sqrt(V(2));
end
f = [-Q1;Q1-Q2];
```

Note how special consideration must be made to prevent volume values becoming negative, which is not possible in this case.
Experimental Testing and Evaluation of Model

- Testing a one-can model. Given an experimentally determined, $K$, repeated tests can be run to determine if time-to-empty is accurately predicted. To fully assess accuracy of the model, the volume needs to be measured over time and compared to simulation predictions.

- Testing a two-can model. One quick test is to begin with top can at a given volume, bottom can empty, and estimate peak level reached in the bottom can and time-to-empty. This is one test to validate the accuracy of the two-can system model.

- Additional validation can be made by measuring volume of each can over time.

NOTE: We use the ‘filling’ of the bottom can as an assessment of the accuracy of the model. This test requires that the flow coefficients for both cans and the initial condition of the top can are all accurately determined.
The one and two-can systems rely on physical modeling of hydraulic system elements, and introduce the application of state space modeling.

The system model is described by two nonlinear differential equations in state space form.

Each can requires its own ‘$K$’ value which must be experimentally determined.

To make predictions about the two volume states of this system, it is necessary to solve the equations using a numerical simulation program.

We can test our model using the bottom can ‘filling’, however if we needed to control this system we would likely require use of sensors to provide direct measurement of the system states.
Orifice flow model

Begin by using mass continuity and the steady Bernoulli equation to derive a relation for the velocity of the fluid exiting at control surface 2 (see right).

In this case,

\[
0 = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot d\vec{A}
\]

\[
\frac{P_1}{\rho} + \frac{V_1^2}{2} + gh_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gh_2
\]

to show that:

\[
V_2 = \sqrt{\frac{2gh}{1 - (A_2/A_1)^2}}
\]

What is \(A_2\)? Approximate by using the jet area. If there is an orifice where the jet area is not equal to the orifice area, we use a contraction coefficient.
Orifice contraction coefficients

The contraction coefficient is defined by,

\[ C_c = \frac{A_{jet}}{A_o} \]

Some common experimentally determined values:
Velocity coefficient

Using the contraction coefficient, our model for the exiting mass flowrate is,

\[ \dot{m} = \rho Q = \rho A_{\text{jet}} V = \rho C_c A_o C_v V_{\text{ideal}} \]

where we’ve used the velocity coefficient,

\[ C_v = \frac{V_{\text{actual}}}{V_{\text{ideal}}} = \text{velocity coefficient} \]

which accounts for friction effects.
Orifice flowrate model

Use the ideal velocity from the Bernoulli equation,

\[ V_{\text{ideal}} = V_2 = \sqrt{\frac{2gh}{1 - (A_2/A_1)^2}} \]

and gauge pressure, \( P \), at the bottom of the can so the flowrate from the orifice is,

\[ Q = \frac{m}{\rho} = \frac{C_c C_v A_o}{[1 - C_c^2 (A_2/A_1)^2]} \sqrt{\frac{2P}{\rho}} = K_o A_o \sqrt{\frac{2P}{\rho}} \]

where \( \rho \) is assumed constant and,

\[ K_o = \frac{C_c C_v}{[1 - C_c^2 (A_2/A_1)^2]^{1/2}} \]

NOTE: Correction

This should have square root
Orifice flow relation

The flowrate relation derived is used extensively for orifice flow predictions in a wide range of applications. For example, it is commonly applied when predicting hydraulic valve flows.

If the pressure can reverse sign, then the relation may need to be generalized to account for sign by using,

$$Q = K_o A_o \text{sgn}(P) \sqrt{2|P|/\rho}$$

where the sgn() function provides sign information and the absolute value of pressure is taken when used in computation.
**Ideal orifice flow constant, $K_{\text{ideal}}$**

In the derived expression for $K_o$, assume ideal contraction and velocity coefficients, $C_c = 1$ and $C_v = 1$ so,

$$K_{o,\text{ideal}} = 1 / \left[ 1 - \left( A_o / A_{\text{can}} \right)^2 \right]^{1/2}$$

and the ideal case flowrate is,

$$Q_{\text{ideal}} = K_{o,\text{ideal}} A_o \sqrt{\frac{2}{\rho}} \sqrt{\frac{|V|}{C}} = K_{o,\text{ideal}} A_o \sqrt{\frac{2}{\rho} \frac{g}{A_{\text{can}}}} \sqrt{|V|} = K_{\text{ideal}} \sqrt{|V|}$$

where the ideal can coefficient is defined by,

$$K_{\text{ideal}} = K_{o,\text{ideal}} A_o \sqrt{2g/A_{\text{can}}}$$

a value calculated from easily measured dimensions.
Example $K_{\text{ideal}}$ calculations

Example calculation of ideal flow coefficient calculations given in different units are shown below.

\[
\begin{align*}
D_{\text{can}} &:= 3\text{-in} \quad D_{o} := \frac{1}{8}\text{-in} \\
A_{\text{can}} &:= \frac{\pi \cdot D_{\text{can}}^2}{4} \quad A_{o} := \frac{\pi \cdot D_{o}^2}{4} \\
K_{\text{ideal}} &= K_{\text{ideal}} \cdot A_{o} \cdot \sqrt{\frac{2 \cdot g}{A_{\text{can}}}} \\
K_{\text{ideal}} &= 1.000001507 \\
K_{\text{ideal}} &= 0.128 \sqrt{\frac{\text{in}^3}{\text{s}}} \\
K_{\text{ideal}} &= 5.192 \times 10^{-4} \sqrt{\frac{\text{m}^3}{\text{s}}} \\
K_{\text{ideal}} &= 0.128 \sqrt{\frac{\text{in}^3}{\text{s}}} 
\end{align*}
\]
Can dimensions for two-can systems

**Two Can Lab – Can Dimensions**

- **D** = inner diameter
- **d** = orifice diameter
- **H** = inside height

<table>
<thead>
<tr>
<th>Can Set</th>
<th>Small Can</th>
<th>Large Can</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( D_2 ) (in)</td>
<td>( d_2 ) (in)</td>
</tr>
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<td>3.007</td>
<td>0.123</td>
</tr>
<tr>
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<td>3.002</td>
<td>0.124</td>
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</tr>
<tr>
<td>5</td>
<td>3.002</td>
<td>0.124</td>
</tr>
</tbody>
</table>

**Note:** All measurements are in inches.