Transfer function form of the model

\[ G_{\theta/v}(s) = \frac{\theta_n}{v_{in}} = \frac{K_{\theta/v} \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

The following slides detail a derivation of this analog meter model both as state space model and transfer function (TF) as shown above.

Why use the TF model? The static experiments provide a measure of the static gain, so only two more parameters are needed – 3 total. Also, the step response looks like a classic 2\textsuperscript{nd} order system response.

The state space model needs 5 parameters, so it is not as convenient to use in this control study.

TF model only needs damping ratio and natural frequency, which can be readily determined in the laboratory.
Analog moving coil meter model

- The meter physical model is presented in two different versions: ‘full model’ and 2\textsuperscript{nd} order.
- The 2\textsuperscript{nd} order model neglects inductance, which is a good assumption for this application.
- The electromechanical (EM) torque can be modeled using a gyrator.
Recall the models shown before in schematic form:

Electrical circuit model

Rotational system

Modeling
The EM torque induced on the moving coil is related to the current – a gyrator model

This slide summarizes the basic force-current relation in a conductor. In a bond graph, this can be modeled by a gyrator, which gives a net relation between force and current. For a motor or for the case of the rotational moving coil, this force is resolved into torque.

The differential force on a differential element of charge, \( dq \), is given by:

\[
\text{where } B \text{ is the magnetic field density, and } i \text{ the current (moving charge).}
\]

It can be shown that the net effect of all charges in the conductor allow us to write:

\[
\text{where } dl \text{ is an elemental length.}
\]

For a straight conductor of length \( l \) in a uniform magnetic field, you can integrate to find the total force:

\[
\text{With angle } \alpha \text{ between the vectors, you can arrive at the desired relation:}
\]

\[
F = (Bl \sin \alpha) \cdot i
\]
Analog moving coil meter bond graph

A bond graph of the meter can take the form shown below. The coil has resistance, $R_m$, and inductance, $L_m$. The needle has moment of inertia, $J_n$, and there is some damping, $B_n$, as well. The spring has stiffness, $K_s$. These are parameters for linear constitutive relations for each of the elements shown in this model. Note, the meter also has an external series resistor that is not shown here, but the value of that resistance can be added to $R_m$.

We seek a mathematical model that relates needle position (equal to spring deflection) to input voltage, $v_{in}$.

This model can be derived from the bond graph, or by application of Newton’s Laws (mechanical side) and KVL (circuit side).

Electrical circuit model

$\begin{align*}
E & \rightarrow v_{in} \\
1 & \downarrow \\
I : L_m & \rightarrow i_m \\
\dot{i}_m & \rightarrow r_m \\
G & \rightarrow \omega_m \\
1 & \downarrow \\
C : K_s^{-1} & \rightarrow \theta_s
\end{align*}$

EM conversion

$\begin{align*}
1 & \downarrow \\
1 & \downarrow \\
1 & \downarrow \\
T_s & \rightarrow \theta_s
\end{align*}$

Rotational system

$\begin{align*}
R : B_n & \rightarrow \omega_B \\
T_B & \rightarrow \theta_s
\end{align*}$

$R_T = R_m + R_s$

See Appendix B for explanation of gyrator model for EM transduction.
Full 3\textsuperscript{rd} order model, with inductance

The state-space model for the meter, including the inductance, is 3\textsuperscript{rd} order.

\[
\begin{align*}
\dot{h}_n &= J_n \omega_n = \text{needle angular momentum} \\
\dot{\lambda}_m &= L_m i_m = \text{flux linkage} \\
\dot{\theta}_n &= \text{angular position of needle/spring}
\end{align*}
\]

3 States:

\[
\begin{align*}
\dot{h}_n &= J_n \omega_n = T_m - K_s \theta_n - B\omega_n \\
\dot{\lambda}_m &= L_m \left( \frac{di_m}{dt} \right) = v_{in} - (R_m + R_s)i_m - v_{in} \\
\dot{\theta}_n &= \omega_n
\end{align*}
\]

3 State equations:

EM gyrator relations:

\[
\begin{align*}
T_m &= r_m i_m \\
v_m &= r_m \omega_m
\end{align*}
\]

Note: the needle and the spring have the same velocity.

Also, can choose either the meter flux linkage or current as the state.
3rd order model state-space equations

In state space form:

\[
\begin{bmatrix}
\dot{\lambda}_m \\
\dot{h}_n \\
\dot{\theta}_n
\end{bmatrix} = \begin{bmatrix}
\frac{-R_T}{L_m} & -\frac{r_m}{J_n} & 0 \\
\frac{r_m}{J_n} & -\frac{B_n}{J_n} & -K_s \\
0 & \frac{1}{J_n} & 0
\end{bmatrix} \begin{bmatrix}
\lambda_m \\
h_n \\
\theta_n
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} v_{in}
\]

Output equation:

\[
y = \theta_n = \begin{bmatrix}
0 & 0 & 1 \\
0 & h_n & 0 \\
0 & \theta_n & 0
\end{bmatrix} + \begin{bmatrix}
0
\end{bmatrix} v_{in}
\]
Analog moving coil meter bond graph – neglect inductance

If we neglect the inductance, we see that the model reduces to second order. Note the change in causality (if you understand bond graphs).

This assumption is reasonable given that we observe a step response in the experiments that looks 2\textsuperscript{nd} order, underdamped.

Now the only states of interest are the needle angle (related to spring deflection) and the needle rotational momentum.

\[ R_T = R_m + R_s \]

See Appendix B for explanation of gyrator model for EM transduction.
2nd order model, neglecting inductance

The mathematical model for the meter, *neglecting inductance*,

\[
\begin{align*}
\text{States:} & \\
& \begin{cases} 
  \dot{h}_n = J_n \omega_n = \text{angular momentum} \\
  \dot{\theta}_n = \text{angular position of needle/spring}
\end{cases} \\
\text{State equations:} & \\
& \begin{cases} 
  \dot{h}_n = T_m - K_s \theta_n - B_n \omega_n \\
  \dot{\theta}_n = \omega_n
\end{cases} \\
\text{EM gyrator relations:} & \\
& \begin{cases} 
  T_m = r_m i_m \\
  v_m = r_m \omega_m
\end{cases} \\
\text{where,} & \\
& i_{in} = i_R = \frac{(v_{in} - v_m)}{(R_m + R_s)}
\end{align*}
\]

The meter current is now determined by the voltage drop, not by the inductor state.
2nd order state-space equations

In state space form:

State equations:

\[
\begin{bmatrix}
\dot{h}_n \\
\dot{\theta}_n
\end{bmatrix} =
\begin{bmatrix}
-\left( B_n + \frac{r_m^2}{R_T} \right) \frac{1}{J_n} & -K_s \\
\frac{1}{J_n} & 0
\end{bmatrix}
\begin{bmatrix}
h_n \\
\theta_n
\end{bmatrix} +
\begin{bmatrix}
\frac{r_m}{R_T} \\
0
\end{bmatrix} v_{in}
\]

\( A \)

Output equation:

\[
y = \theta_n =
\begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{h}_n \\
\dot{\theta}_n
\end{bmatrix} +
\begin{bmatrix}
0
\end{bmatrix} v_{in}
\]

\( C \)
Let’s convert this into a 2\textsuperscript{nd} order ODE equation

First, consider just the first equation and write it in terms of the angle:

\[ h_n = J_n \omega_n = J_n \dot{\theta}_n \Rightarrow \dot{h}_n = J_n \ddot{\theta}_n \]

Substitute into the momentum equation:

\[
\Rightarrow J_n \ddot{\theta}_n + \left( B_n + \frac{r_m^2}{R_T} \right) \frac{1}{J_n} \left[ J_n \dot{\theta}_n \right] + K_s \theta_n = \frac{r_m}{R_T} v_{in}
\]

Remember that the angle and angular velocity are related, so write in terms of the angle. This gives the 2\textsuperscript{nd} order ODE we want:

\[
\Rightarrow \ddot{\theta} + \left( B_n + \frac{r_m^2}{R_T} \right) \frac{1}{J_n} \dot{\theta} + \left( \frac{K_s}{J_n} \right) \theta = \frac{r_m}{J_n R_T} v_{in} = \left( \frac{K_s}{J_n} \right) \left( \frac{r_m}{K_s R_T} v_{in} \right)
\]
Relate to the static gain measured in lab

\[
\ddot{\theta}_n + \left( B_n + \frac{r_m^2}{R_T} \right) \frac{1}{J_n} \dot{\theta}_n + \left( \frac{K_s}{J_n} \right) \theta_n = \left( \frac{K_s}{J_n} \right) \left( \frac{r_m}{K_s R_T} \right) v_{in}
\]

For a constant input voltage, the steady-state angle (equilibrium) is found by making the derivative terms zero,

\[
\ddot{\theta}_n = 0 \quad \Rightarrow \quad \theta_{nss} = \left[ \frac{r_m}{K_s R_T} \right] v_{in} = K_{\theta/v} v_{in}
\]

So, if you want a certain angle, you simply apply, \( v_{in} = K_{v/\theta} \theta_{desired} \)

Works well if parameters are known, remain constant, and dynamic effects are not significant.
Now write in transfer function form

Start in the standard form: \[ \ddot{\theta}_n + 2\zeta \omega_n \dot{\theta}_n + \omega_n^2 \theta_n = \omega_n^2 u(t) \]
\[ u(t) = K_{\theta/v} v_{in} \]

Transform to s-domain, and solve for angle-to-voltage relation:
\[ \frac{\theta_n}{v_{in}} = \frac{K_{\theta/v} \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

You can see this model returns the static gain relation when you make \( s \) go to zero (i.e., steady-state).

So, all we need is the damping ratio and the natural frequency to parameterize this dynamic model. These can be found either from the physical parameters or by experimentally determining the values in the lab. We’ll do the latter.