Modeling and Experimentation:
Mass-Spring-Damper System Dynamics

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This lab is meant to build and strengthen understanding of mass-spring-damper system dynamics, a familiar model often introduced in a first physics course.

The system dynamics are analogous to those in many types of physical systems.

Understanding the model and system response is critical to designing all types of engineering systems and devices, and for understanding the specifications of systems.

This insight also guides the design of physical experiments for systems that have second order system characteristics.

These slides review concepts and principles needed for planned laboratory study with a mass-beam system.
Ubiquitous mass-spring-damper model

Consider two common configurations of the mass-spring-damper model.

*Fixed-base* configuration, spring and damper in parallel.

\[ F(t) \]
\[ p = m V_m \]
\[ F_k = k x_k \]

*Base-excited* configuration, spring and damper in parallel, motion input at base.

\[ F_b = b V_b \]

When a parameter like \( k \) or \( b \) is indicated, it usually implies that a *linear* constitutive law is implied for that model element.
Mass-spring-damper models of practical systems

Mass-spring-damper models are used to study many practical problems in engineering.

*Fixed-base* configuration: mechanical structures, buildings, etc.

*Base-excited* configuration: vehicle suspension, seismic sensors

In addition to design and analysis of engineering and physical systems, these models provide insight into design of physical experiments. Analysis of these models examines both unforced and forced response.
Using mass-spring-damper models

1. Use these models to represent a wide range of practical situations, not just translation but rotation. The models have analogies in all other energy domains.

2. Recognize ‘forcing’ in each case: force $F(t)$ on mass for fixed-based compared with velocity $\dot{y}(t)$ at the base for base-excited system.

3. Unforced response. Some problems are concerned with the system responding to initial conditions and no forcing (i.e., $F(t) = 0$, $y(t) = 0$), so the transient response and the eventual rest condition is of interest.

4. Forced response. If there is forcing, there may be a need to understand transient changes and then the ‘steady’ operation under forcing. Types include: step, impulse, harmonic, random.
Example: unbalanced fan on support base

An unbalanced fan is mounted on a support base, and the model to study induced vibration is illustrated below. This is a fixed-base configuration.

An additional modeling issue in this problem is coming up with the forcing function, $F(t)$, which would come from understanding how the eccentricity in the fan impeller induces a force on the total fan mass. Note an assumption can also made that the fan only vibrates in the vertical direction, if the base structure is very stiff laterally. All of these decisions are part of the modeling process.
Example: motion sensor (seismic sensor)

An example of a system that is modeled using the based-excited mass-spring-damper is a class of motion sensors sometimes called seismic sensors. The spring and damper elements are in mechanical parallel and support the ‘seismic mass’ within the case. The case is the base that is excited by the input base motion, $y(t)$. 
Example: suspended mass moving over a surface

The diagram below is uses a base-excited configuration to model a mass moving over a surface. The spring and damper elements might represent, for example, the tire contacting the ground as the vehicle moves in the $x$ direction. Vibration of the mass is in the $z$ direction.

An automotive suspension model like this would represent only a quarter of the vehicle, and there would be another stage that represents the actual suspension. Note how you model the base-motion by factoring how fast you move over a ground profile, $z_s(x)$, a function of distance traveled, $x$. 

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Consider the fixed-base system below.

Applying Newton’s law to a free-body diagram of the mass,

\[ m \ddot{x} = \sum F = -F_b - F_k + F, \]

\[ m \dddot{x} = -b \dot{x} - kx + F, \]

Rearrange to derive a 2nd order ODE,

\[ m \ddot{x} + b \dot{x} + kx = F. \]

Now, in standard form,

\[ \ddot{x} + \begin{bmatrix} \frac{b}{m} \end{bmatrix} \dot{x} + \begin{bmatrix} \frac{k}{m} \end{bmatrix} x = u \]

where \( u = F/m, \omega_n \) is the undamped natural frequency, and \( \zeta \) is the damping ratio.

From the undamped natural frequency, define the undamped natural period,

\[ T_n = \frac{2\pi}{\omega_n}. \]
**Undamped harmonic motion, \( \zeta = 0 \)**

For the unforced case, \( F = 0 \), and undamped, \( b = 0 \), case,

\[
x = x_o \cos(\omega_n t) + \frac{\dot{x}_o}{\omega_n} \sin(\omega_n t)
\]

where \( x_o \) is the initial displacement and \( \dot{x}_o \) the initial velocity of the mass. This solution gives \( x \) for all time given only the initial conditions and the natural frequency, which in this case is \( \omega_n = \sqrt{k/m} \).

**NOTE:** Relate this to any other ‘linear oscillator’ system that has the 2nd order ODE,

\[
\ddot{x} + \omega_n^2 x = 0
\]

For example, the simple or compound pendulum in small motion follows this form, and we can see how \( \omega_n \) in that case would contain parameters such as length, gravity, mass.
**Undamped harmonic motion**

If you give the mass an initial displacement $x_0$ and release from rest ($\dot{x}_0 = 0$), then this case predicts it would oscillate indefinitely,

$$x = x_0 \cos(\omega_n t)$$

Given $x(t)$, the velocity and acceleration are readily found:

$$v = -x_0 \omega_n \sin(\omega_n t)$$

$$a = +x_0 \omega_n^2 \cos(\omega_n t)$$

**NOTE:** The undamped natural frequency directly influences peak values of velocity and acceleration.
What about the effect of gravity?

The axis of mass motion may be in line with the gravity vector, so sometimes it is necessary to consider the effect of the force due to gravity, $mg$.

In vibration problems, however, especially when all the elements follow linear constitutive laws (i.e., spring, $F = kx$, damper, $F = bV$), the effect of gravity will not influence the dynamics. Only the equilibrium position of the mass will be changed.

If you had a spring that was not linear, then it is important to consider gravity because vibrational motion will be influenced by the spring behavior in a nonlinear manner.

**Bottom line:** If system is linear, use gravity to find where your system ‘sits’ initially, but then the linear solutions discussed here apply directly for motion about that point. If nonlinear, you may need to follow other approaches such as linearizing about the equilibrium, which is found by considering gravity. Direct numerical simulation of the system is often an alternative.
Consider the damped cases now, $\zeta \neq 0$

The special \textit{undamped} case has been described. For systems where $b \neq 0$, the damping ratio will not be zero. Solutions for these cases are classified by $\zeta$, and a system is:

- \textit{underdamped} if $\zeta < 1$,
- \textit{overdamped} if $\zeta > 1$,
- \textit{critically damped} if $\zeta = 1$

The solutions are known for these cases, so it is worthwhile formulating model equations in the standard form,

$$\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = u(t)$$

Detailed derivations can be found in system dynamics, vibrations, circuits, etc., type textbooks. Selected excerpts will be posted on the course log for reference.
Underdamped motion, $\zeta < 1$

If you give the mass an initial displacement $x_o$ and initial velocity $\dot{x}_o$, the solution for the response, $x$, is,

$$
x = e^{-\zeta \omega_n t} \left[ \frac{\dot{x}_o + \zeta \omega_n x_o}{\omega_d} \sin(\omega_d t) + x_o \cos(\omega_d t) \right], \ 0 < \zeta < 1
$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 2\pi / T_d$ is the damped natural frequency, and $T_d$ the damped natural period.

Given $x(t)$, the velocity and acceleration can be found by differentiation.

**NOTE:** The damped natural frequency is dependent on both the undamped natural frequency and the damping ratio.
Pre-Lab 1. Example mass-spring-damper parameter calculations

For a mass-spring-damper system that has the values $m = 5 \text{ kg}$, $k = 200 \text{ N/m}$, and $b = 4 \text{ N*s/m}$

(a) calculate the natural frequency,

(b) calculate the damping ratio and confirm that this system is underdamped,

(c) calculate the damped natural frequency, $\omega_d$, and explain the difference from undamped natural frequency, $\omega_n$. 
A beam-mass system

A mass-spring-damper system model can be used to model a flexible cantilevered beam with an affixed mass on the end, as shown below.

A cantilevered beam can be modeled as a simple translational spring with indicated stiffness.

With relatively small tip motion, the beam-mass approximates a mass-spring system reasonably well.

A fixed-base model is assumed, with $F(t) = 0$.

Damping arises from material damping in the beam and from movement of the beam and mass in surrounding air.
Modeling the beam-mass system

Say you took a beam in lab with a mass mounted at the tip. If you deflected it a known amount and then released it from rest, how well could you predict:

1. the displacement amplitude over time?
2. the velocity amplitude over time?
3. the acceleration amplitude over time?

Assume you are specifically asked to predict the number of oscillations the system will experience for a given test, the peak values for each oscillation, and how long it takes to stop.

From the response equation, you know you need natural frequency, damping ratio, and the initial deflection to estimate $x$ over time.
Pose this as a design problem

Let’s discuss how this model could be used in a practical design problem.

If you were asked to explore a design space and to select a beam and attached mass so the system would meet certain specifications, how would you go about finalizing this design?

1. You don’t have a model for damping in the system, $b$, to estimate $\zeta$.

2. Even for known mass attached to the beam, the beam also has some mass, $m_b$. Can this mass be ignored when you estimate $\omega_n$?

3. You’d need to know the initial condition to within some level of precision dictated by the requirements of the problem.

⇒ It should be clear that even for such a simple system, you need to design and run some physical experiments.
Typical laboratory objectives

Attach an accelerometer and any additional mass to the end of a cantilevered beam to form a beam-mass system. For this system, determine:

1. the undamped natural frequency, \( \omega_n \),
2. effective system damping ratio, \( \zeta \)
3. total effective mass of the beam-mass system

The total effective mass is the sum of any attached mass, accelerometer mass, and some unknown fraction of the beam mass. This setup can then be used to understand the beam-mass system.
Determine practical limits on undamped natural frequency

We can calculate upper and lower bounds on the undamped natural frequency in the following way. Assume \( k_b \) is the beam stiffness, \( m \) is the total of any attached mass (including total sensor mass), and \( m_b \) is the total beam mass.

**Case 1:** Assume beam is *massless*. In this case, the undamped natural frequency is,

\[ \omega_{n1} = \sqrt{\frac{k_b}{m}}. \]

**Case 2:** Assume all the beam mass is concentrated at the tip. In this case, the undamped natural frequency is,

\[ \omega_{n2} = \sqrt{\frac{k_b}{(m + m_b)}}. \]

The *actual* undamped natural frequency has to lie between these two values,

\[ \omega_{n2} \leq \omega_{\text{actual}} \leq \omega_{n1} \]

The actual value will require some fraction of the beam mass less than 1.0. If we could measure undamped natural frequency (i.e., or period) directly, then this fractional value of \( m_b \) could be measured. First, consider at least one theoretical estimate.
Theoretical estimate of undamped natural frequency

From appendix in Den Hartog [1] (see handout on course log), the undamped natural frequency for a beam-mass system can be estimated as follows:

Note that a fractional beam mass of $0.23m$ is used in this formula, where $m$ in this formula is the total beam mass (cf. our notation, $m_b$).

This theoretical estimate is based on Rayleighs method, which is described in a handout from Thomson (Ch. 2) [2]. A derivation for the beam-mass case is presented in a handout “Estimating the mass-beam natural frequency using Rayleigh’s method”. Both handouts are posted on the course log.
Pre-Lab 2. Estimate limits and theoretical $\omega_n$

Before you run any experiments, you can calculate expected values using the methods just described. For this pre-lab problem, submit answers to the following:

1. You should have all the parameters for the aluminum cantilevered beam used in the strain gauge lab. Assume that an accelerometer with a total mass of 46 grams is placed at the tip (define this distance). Calculate the upper and lower bounds on the undamped natural frequency for this system. Report both in units of rad/sec and in Hertz (Hz). The relation is $\omega \text{ (rad/sec)} = 2\pi f_n$, where $f_n$ is the undamped natural frequency in Hz.

2. Use the estimate from Den Hartog (equation 21) and calculate the mass fraction, $0.23m_b$, where $m_b$ is your total beam mass. Using this value and the value of the accelerometer mass, what is the undamped natural frequency this estimate would provide in units of Hertz (Hz)?
Experimental determination of $\omega_n$

We need to validate the models, so laboratory measurement of undamped natural frequency is needed. If we can measure the motion of the beam tip after it has been deflected and released from rest, then the measured period can be used to estimate the damped natural period, $T_d$.

However, only in the case of negligible damping can the undamped natural period be approximated using the measured period.

Since the damped period is given by,

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}},$$

a measured value of the period can be used to estimate $\omega_n$ only if $\zeta$ has first been determined. The determination of $\zeta$ is considered next.
Experimental determination of damping (1)

From the model for underdamped response of an *unforced* second order system presented earlier,

\[ x = e^{-\zeta \omega_n t} \left[ \frac{\dot{x}_o + \zeta \omega_n x_o}{\omega_d} \sin(\omega_d t) + x_o \cos(\omega_d t) \right], \quad 0 < \zeta < 1 \]

which can be expressed,

\[ x(t) = A_o e^{-\zeta \omega_n t} \cos [\omega_d t - \phi] \]

where,

\[ A_o = \left[ \left( \frac{\dot{x}_o + \zeta \omega_n x_o}{\omega_d} \right)^2 + x_o^2 \right]^{1/2} \quad \text{and} \quad \tan \phi = \frac{\dot{x}_o + \zeta \omega_n x_o}{\omega_d x_o} \]

The important result here is that the amplitude of the response follows a *decaying exponential* that is a function of initial conditions, \( \zeta \), and \( \omega_n \).
Experimental determination of damping (2)

The amplitude decay can be determined using peak values measured every $T_d$ seconds every cycle, $n$. The amplitude ratio between successive peaks is then,

$$\frac{A_n}{A_{n+1}} = \exp \left[ \zeta \omega_n (t_{n+1} - t_n) \right]$$

and the period between peaks is,

$$(t_{n+1} - t_n) = T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}}.$$

Now, taking log of both sides of the amplitude ratio relation,

$$\ln \left[ \frac{A_n}{A_{n+1}} \right] = \left[ \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \right] \cdot n$$

This shows that log of the amplitude ratios (the log decrement) is \textit{linearly} related to the cycle number, $n$, by a factor that is a function of $\zeta$. 

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Consider two decay envelopes

**Exponential decay:**

An envelope that takes on an exponential shape may suggest linear damping is dominant, while an envelope on the right with linear decay is likely not linear. These insights must be *quantified* by measurement.
Log decrement quantifies the damping

The relation revealed from the log decrement data will suggest if a system has dominant linear damping.

Case (a) on the left indicates linear damping, while case (b) is for a system with Coulomb (nonlinear) damping, for which a constant $\zeta$ is not defined.

Note that the slope of the log decrement plot is, $\beta = \left\lfloor \frac{2\pi \zeta}{\sqrt{1-\zeta^2}} \right\rfloor$, from which $\zeta$ can be found.

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Pre-Lab 3. Example estimation of system damping using logarithmic decrement

A LabVIEW VI (to be provided) runs a simulation of a mass-spring-damper system with the specified parameters. Experiment with this VI, which shows how a damping ratio ($\zeta$) can be determined from the logarithmic decrement. Do the following:

(a) Study the simulation code.

(b) Study the code to show how a Peak Detector VI built into LabVIEW can be used to find the peaks of a decaying motion signal, in this case from a simulation.

(c) Use the parameter values from Pre-Lab 2, for which values of $\omega_n$ and $\zeta$ were found. Run the VI and confirm that the value of $\zeta$ is extracted. In the simulation, use $x_0 = 0.1$ m, and $\dot{x}_0 = 0$ (release from rest).

A screen shot showing all parameters used and the value of $\zeta$ determined should be submitted.
Lab Objectives

1. Determine reasonable values for the beam stiffness, added mass (accelerometer) and beam mass.

2. With an accelerometer mounted on the end of a strain-gauged beam, a beam-mass system will be created. Experimentally determine the damped natural period and the system damping ratio.

3. Use your measured damping ratio to finalize estimates for the undamped natural frequency and the total (effective) mass.

4. Use your estimate of the total (effective) mass to identify the fraction of the beam’s mass that is effectively added to the tip mass. Compare this quantity with that recommended in Den Hartog’s table (see item 21 in Den Hartog formulas) [1].
Pre-Lab 4. Lab procedure

Sketch out lab procedures for completing the proposed lab objectives. The TA will review your procedures and provide guidance/feedback, but you will be expected to complete the lab work using your own approach with minimal guidance.
Summary

- Mass-spring-damper system models help understand a wide range of practical engineering problems and design of motion sensors.
- You can understand the underlying design of many types of sensors such as accelerometers by understanding 2nd order system dynamics.
- In this lab, we will use an accelerometer to measure acceleration. A more detailed study of how these devices work may be made in a later lab. A related lecture provides an overview of accelerometers.
Pre-Lab 5. Understanding accelerometer basics

Study the general specifications for the CXL04LP3 model accelerometer we will use in this lab. Explain what is meant by sensitivity, input range, bandwidth, and zero g output. Summarize the values for these specifications as listed in the datasheet provided on the course log.

**NOTE:** A short discussion/lecture on accelerometer concepts will be posted and should be reviewed.
Lab Evaluation

The TA will ask you to evaluate your lab work as follows. One or more of these may be made optional by the TA.

1. Explain why it is never really possible to directly measure undamped natural frequency.

2. Report the values you determined for system damping ratio, the damped and undamped natural frequencies, and for the total (effective) mass and the fractional beam mass.

3. Capture the acceleration of your tip using an accelerometer and/or a displacement sensor (e.g., a laser displacement sensor). For a given (known) initial condition, release the tip from rest and capture the signal. Use the response model to calculate peak values at the first three cycles (after release). How well does your model predict acceleration and/or displacement?

4. Final evaluation of your method - the TA will add an additional lumped mass to your system. You will be asked to determine that mass using your methodology. One or more iterations may be requested.
References
