Mathematical Preliminaries

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Theory of Elasticity notes.

1) Indicial Notation

Indicial notation is a shorthand scheme whereby a whole set of real numbers is indicated by a single symbol with subscripts. For example, the 3 real numbers \( a_1, a_2, a_3 \) are denoted by \( a_i \).

**Range Convention:** Unless otherwise specified, all subscripts have the range 1, 2, 3.

Thus, \( a_{ij} \) stands for the 9 real numbers

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

Such a rectangular array is called a **matrix** and is usually set off in brackets [ ], double bars \( | | \), or parentheses ( ). We will write

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

The three elements \( a_{ij} \) constitute the first row, while \( a_{i1} \) is the first column, etc.
In general, a symbol \( a_{ij} \ldots k \) with \( N \) unspecified, distinct subscripts represents \( 3^N \) real numbers. It should be clear that \( a_j \) and \( a_k \) represent the same three numbers as \( a_i \).

Addition, multiplication, and equality of indexed symbols is defined in the natural way. E.g.,

\[
a_i + b_i \text{ denotes the 3 numbers } a_1 + b_1, a_2 + b_2, a_3 + b_3;
\]

\[
a_i b_j \text{ denotes the 9 numbers } a_1 b_1, a_1 b_2, a_1 b_3,
a_2 b_1, a_2 b_2, a_2 b_3,
a_3 b_1, a_3 b_2, a_3 b_3
\]

(The multiplication of two symbols with distinct subscripts is called \textit{outer} multiplication);

\[
\lambda a_i \text{ denotes the 3 numbers } \lambda a_1, \lambda a_2, \lambda a_3 \ (\lambda \in \mathbb{R})
\]

(The multiplication of a symbol without subscripts and one with subscripts is called \textit{scalar} multiplication);

\[
a_i = b_i \text{ stands for the 3 equations } a_1 = b_1, a_2 = b_2, a_3 = b_3.
\]

The operation of subtraction is built up from addition and scalar multiplication; e.g.,

\[
a_i - b_i = a_i + (-1)b_i.
\]

Since we are dealing with real numbers, it is evident that the above operations obey certain rules. E.g.,

\[
a_i + b_i = b_i + a_i \quad \text{commutative rule;}
\]

\[
a_i (b_i + c_i) = (a_i + b_i) + c_i \quad \text{associative rule;}
\]

\[
a_i (b_k c_m) = (a_i b_k) c_m \quad \text{associative rule;}
\]

\[
a_j (b_k + c_k) = a_j b_k + a_j c_k \quad \text{distributive rule;}
\]

etc.

It should be obvious that if
\[ a_i = b_i, \]
then
\[ \lambda a_i = \lambda b_i \]
and
\[ a_i c_j = b_i c_j \]
No meaning is assigned to expressions such as
\[ a_i + b_j \text{ or } c_{ij} = a_k. \]
A symbol \( a_{ij...m...n...k} \) is said to be symmetric w.r.t. the indices \( m \) and \( n \) if
\[ a_{ij...m...n...k} = a_{ij...n...m...k}; \]
it is antisymmetric or skewsymmetric w.r.t. \( m \) and \( n \) if
\[ a_{ij...m...n...k} = -a_{ij...n...m...k}. \]
The obvious identity
\[ a_{ij} = \frac{1}{2} (a_{ij} + a_{ji}) + \frac{1}{2} (a_{ij} - a_{ji}) \]
shows that a 2-index symbol \( a_{ij} \) can be written as the sum of a symmetric and skewsymmetric symbol.
The symmetric and skew parts of \( a_{ij} \) are denoted by \( a_{(ij)} \) and \( a_{[ij]} \), respectively; i.e.,
\[ a_{ij} = a_{(ij)} + a_{[ij]}, \]
where
\[ a_{(ij)} = \frac{1}{2} (a_{ij} + a_{ji}), \quad a_{[ij]} = \frac{1}{2} (a_{ij} - a_{ji}). \]

2) The Summation Convention

Summation Convention: If an unspecified subscript appears twice in the same monomial, then summation over that subscript from 1 to 3 (or whatever range is appropriate) is implied. Thus,
\[ a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}, \]
\[ a_{ij} b_j = \sum_{j=1}^{3} a_{ij} b_j = a_{11} b_1 + a_{12} b_2 + a_{13} b_3, \]

etc. The summation convention is suspended by writing "no sum" or by underlining one of the repeated subscripts.

Clearly, \( a_{ii} = a_{ji} = a_{ki} \), etc.; and \( \therefore \) repeated subscripts are dummy subscripts.

Unspecified subscripts which are not repeated are called free subscripts.

The process of setting two free subscripts in a monomial equal and summing is called contraction. E.g., \( a_{ij} b_j c_i \) is obtained from \( a_{ij} b_j c_k \) by contracting on \( i \) and \( k \).

The operation of outer multiplication of two symbols followed by contraction w.r.t. one subscript from each symbol is called inner multiplication. E.g., \( a_{jk} b_{rk} \) is an inner product.

Clearly, we can contract in a given equation and preserve the equality. E.g., suppose

\[ a_{ij} = b_{ij} \]

Then in particular \( a_{11} = b_{11}, a_{22} = b_{22}, a_{33} = b_{33}, \) and addition yields

\[ a_{11} + a_{22} + a_{33} = b_{11} + b_{22} + b_{33}; \]

i.e., \( a_{ii} = b_{ii} \).

3) **Kronecker's Delta and the Alternating Symbol**

**Definition 3.1** Kronecker's delta is the 2-index symbol \( \delta_{ij} \) defined by

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j \text{ (no sum)}, \\
0 & \text{if } i \neq j 
\end{cases}
\]

Thus, \( \delta_{11} = \delta_{22} = \delta_{33} = 1, \)
\( \delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0. \)

**Theorem 3.1**
(i) $\delta_{ij} = \delta_{ji}$ (Kronecker's delta is symmetric)
(ii) $\delta_{ii} = 3$
(iii) $\delta_{i} = 1$

Theorem 3.2
(i) $\delta_{ij}a_j = a_i$, $\delta_{ij}a_j = a_j$ (Kronecker's delta is often called the substitution operator)
(ii) $\delta_{ij}a_{ik} = a_k$
(iii) $\delta_{ij}a_{ij} = a_{ii}$
(iv) $\delta_{ij}\delta_{ij} = 3$.

Definition 3.2 If each element of the list $ijk$ denotes one of the integers from 1 to 3 (inclusive) and if each of the integers from 1 to 3 appears once and only once in the list, then the list $ijk$ is called a permutation of the integers from 1 to 3.

Definition 3.3 When in a permutation an integer precedes a smaller integer, the permutation is said to contain an inversion.

Clearly, the total number of inversions in a permutation is found by counting the number of smaller integers following each integer of the permutation. Thus 3,2,1 has 3 inversions since 3 is followed by 2 and 1, and 2 is followed by 1.

Definition 3.4 A permutation is even or odd according as the total number of inversions in it is even or odd.

Definition 3.5 The alternating symbol is the 3-index symbol $\varepsilon_{ijk}$ defined by

$$
\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{if } ijk \text{ is not a permutation of } 1,2,3 \end{cases}
$$

Thus,
$$
\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1,
\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1,
\varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} = ... = 0.
$$

Theorem 3.3 The alternating symbol is antisymmetric w.r.t. any two of its indices; e.g.,
\[ \varepsilon_{ijk} = -\varepsilon_{jik}. \]

4) **Determinants**

**Definition 4.1** Given a square matrix \( [a_{ij}] \), its **determinant** is the number

\[
\det [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\]

It should be clear that \( i \) and \( j \) on the l.h.s. of the definition of the determinant are dummy subscripts.

**Theorem 4.1**

(i) If all the elements of a row (column) of \( [a_{ij}] \) are zero, then \( \det [a_{ij}] = 0 \).

(ii) If the corresponding elements of two rows (columns) of \( [a_{ij}] \) are equal, then \( \det [a_{ij}] = 0 \).

(iii) If \( [b_{ij}] \) is formed from \( [a_{ij}] \) by interchanging two rows (columns), then \( \det [b_{ij}] = -\det [a_{ij}] \).

(iv) If \( [b_{ij}] \) is the same as \( [a_{ij}] \) except that all the elements of a row (column) of \( [b_{ij}] \) are \( \lambda \) times the corresponding elements of the corresponding row (column) of \( [a_{ij}] \), then \( \det [b_{ij}] = \lambda \det [a_{ij}] \).

(v) If each element of the \( k^{th} \) row (column) of \( [a_{ij}] \) is expressed as the sum of two terms, then \( \det [a_{ij}] \) may be expressed as the sum of the determinants of two matrices the elements of whose \( k^{th} \) rows (columns) are the corresponding terms of the \( k^{th} \) row (column) of \( [a_{ij}] \). All other rows (columns) are the same throughout.

(vi) If \( [b_{ij}] \) is formed from \( [a_{ij}] \) by adding a multiple of one row (column) to a different row (column), then \( \det [b_{ij}] = \det [a_{ij}] \).

(vii) If \( [b_{ij}] \) is the transpose of \( [a_{ij}] \), i.e., \( b_{ij} = a_{ji} \), then \( \det [b_{ij}] = \det [a_{ij}] \).

**Theorem 4.2** Suppose that the elements \( a_{ij} \) are all differentiable functions of the real variable \( s \) and write
\[
\frac{d a_{ij}}{d s} = a'_{ij}
\]
then
\[
\frac{d}{d s} \det[a_{ij}] = \left| \begin{array}{ccc} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{array} \right| + \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right| + \left| \begin{array}{ccc} a'_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a'_{31} & a_{32} & a_{33} \end{array} \right|
\]

\textbf{Theorem 4.3}
\[
\varepsilon_{ijk} a_{ip} a_{jq} a_{kr} = \det[a_{mn}] \varepsilon_{pqr}
\]
\[
\varepsilon_{ijk} a_{ip} a_{jq} a_{kr} = \det[a_{mn}] \varepsilon_{pqr}
\]

\textbf{Theorem 4.4}
\[
\varepsilon_{ijk} \varepsilon_{pqr} \det[a_{mn}] = \left| \begin{array}{ccc} a_{ip} & a_{iq} & a_{ir} \\ a_{jp} & a_{jq} & a_{jr} \\ a_{kp} & a_{kq} & a_{kr} \end{array} \right|
\]

\textbf{Theorem 4.5}
\[
(i) \quad \varepsilon_{ijk} \varepsilon_{pqr} = \left| \begin{array}{ccc} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{array} \right|
\]
\[
(ii) \quad \varepsilon_{ijk} \varepsilon_{ipq} = \delta_{pj} \delta_{kr} - \delta_{pi} \delta_{kq}
\]
\[
(iii) \quad \varepsilon_{ijk} \varepsilon_{jqr} = 2 \delta_{kr}
\]
\[
(iv) \quad \varepsilon_{ijk} \varepsilon_{jik} = 6
\]

\textbf{5) The Inverse of a Matrix Linear, Algebraic Equations}

\textbf{Definition 5.1} Given a square matrix \([a_{ij}]\) any matrix \([a^{-1}_{ij}]\) with the properties that
\[
a_{ij}^{-1} a_{kj} = a_{ki} a_{kj}^{-1} = \delta_{ij}
\]
is called an \textit{inverse} of \([a_{ij}]\).

\textbf{Theorem 5.1} A square matrix can have at most one inverse; i.e., the inverse, if it exists, is unique. Thus it makes sense to speak of \textit{the} inverse of a matrix.
Theorem 5.2 $\det[a_{ij}] \neq 0$ is a necessary condition for $[a_{ij}]$ to have an inverse.

Theorem 5.3 $\det[a_{ij}] \neq 0$ is a sufficient condition for $[a_{ij}]$ to have an inverse.

In fact, if $\det[a_{ij}] \neq 0$, then

$$a_{ij}^{-1} = \frac{1}{2\det[a_{rs}]} \varepsilon_{pqrs} \varepsilon_{mn} a_{mp} a_{nq}$$

Theorem 5.4 Let $\det[a_{ij}] \neq 0$. Then the linear equations

$$a_{ij} x_j = b_j$$

have the unique solution

$$x_i = a_{ij}^{-1} b_j = \frac{1}{2\det[a_{rs}]} \varepsilon_{pqrs} \varepsilon_{mn} a_{mp} a_{nq} b_j$$

Theorem 5.5 If $\det[a_{ij}] \neq 0$, then the only solution of the homogeneous equations

$$a_{ij} x_j = 0$$

is the trivial solution, $x_i = 0$.

Theorem 5.6 A necessary and sufficient condition that the homogeneous equations

$$a_{ij} x_j = 0$$

have a nontrivial solution is that $\det[a_{ij}] = 0$.

6) Vectors

Let $X$ denote a rectangular Cartesian coordinate frame in 3-dimensional euclidean point space $\varepsilon$. Let $a$ denote an arbitrary vector (directed line element) in $\varepsilon$. The projections of $a$ on the $X_1$, $X_2$, $X_3$ axes are denoted by $a_1$, $a_2$, and $a_3$, respectively. More precisely, if the point $P$ with coordinates $P_i$ w.r.t. $X$ is at the tail of $a$ and the point $Q$ with the coordinates $Q_i$ w.r.t. $X$ is at the head of $a$, then

$$a = PQ, \quad a_i = Q_i + P_i.$$ 

The projection $a_i$ is positive or negative according as the vector $a$ points in the positive or negative $X_i$-direction.
Definition 6.1  The rectangular Cartesian components of $a$ w.r.t. $X$ are the projections $a_i$.

Next we recall some facts from Euclidean geometry. The magnitude or length of $a$ is the number

$$|a| = \sqrt{a_i a_i} \quad \text{(all square roots positive)}$$

The cosine of the angle between $a$ and the $X_i$-axis is

$$\frac{a_i}{|a|}$$

Thus, the components of a vector determine its length and direction—and vice versa.

Theorem 6.1  Two vectors $a$ and $b$ are equal (same magnitude and direction) iff their components are equal, i.e. $a_i = b_i$.

Of course, given a vector $a$ and a number $\lambda$, $\lambda a$ is the vector whose length is $|\lambda| |a|$ and whose direction is the same or opposite as $a$ according as $\lambda > 0$, or $\lambda > 0$.

Theorem 6.2  The components of $\lambda a$ are $\lambda a_i$.

Vectors add according to the parallelogram rule and as was the case with scalar multiplication this is consistent with indicial notation.

Theorem 6.3  The components of $a + b$ are $a_i + b_i$.

Suppose that $e_i$ is a unit vector in the $X_i$ direction. Then if $a_i$ are the components of $a$, we can put the preceding operations together to write

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 = a_i e_i$$

Some authors reserve the name component for the vectors, $a_i e_i$, and then they call $a_i$ the scalar components.
Theorem 6.4  The zero or null vector $0$ is the vector with zero length.
Moreover, $a = 0$ iff $a_i = 0$.

7) Change of Coordinate Frame
In our characterization of vectors by their components, the frame $X$ played such an important role that we must ask "What if we had used a different frame?". Accordingly, let $X'$ denote any other rectangular Cartesian coordinate frame for $\mathcal{E}$, and let $\{e_i'\}$ be the associated set of unit vectors.

Letting $\lambda_{ij}$ be the cosine of the angle between the $X_i$-axis and the $X_j$-axis, we have

$$e_i' = \lambda_{ij} e_j \quad (7.1)$$
and

$$e_i = \lambda_{ij} e_i' \quad (7.2)$$

Now let $a$ be an arbitrary vector with components $a_i$ and $a_i'$ w.r.t. $X$ and $X'$. Then

$$a = a_i e_i \quad (7.3)$$
and

$$a = a_i' e_i' \quad (7.4)$$

Substituting for $e_i$ in (7.3) and (7.2), we have

$$a = a_i \lambda_{ij} e_j' \quad (7.5)$$

but from (7.4),

$$a = a_i' e_i' = a_j' e_j' \quad (7.6)$$

From (7.5) and (7.6),

$$a - a = a_j' e_j' - \lambda_{ji} a_i' e_j' = \left(a_j' - \lambda_{ji} a_i' \right) e_j' = 0.$$  

Similarly, substituting for $e_i'$ in (7.4) from (7.1), we obtain
\[ a_i = \lambda_j a'_j \]

Summing up, we have

**Theorem 7.1** The transformation rules for the rectangular Cartesian components of a vector are

\[ a'_i = \lambda_j a'_j, \quad a_i = \lambda_j a'_j. \]

Here, \( a = a e_i = a'_i' \) and \( \lambda_{ij} \) is the cosine of the angle between \( e'_i \) and \( e_j \).

We now see that to characterize a vector by its components w.r.t. some frame is no essential restriction because we can find its components in any other frame via Theorem 7.1.

The direction cosines \( \lambda_{ij} \) must satisfy certain relationships since the vectors \( e_i \) are mutually orthogonal and of unit length as are \( e'_i \). To find these, we let \( a \) be an arbitrary vector and use the transformation rules to get

\[ a_i = \lambda_j a'_j = \lambda_j \lambda_{jk} a_k \]

But

\[ a_i = \delta_{ik} a_k, \]

and

\[ (\lambda_j \lambda_{jk} - \delta_{ik}) a_k = 0. \quad (7.7) \]

Now (7.7) must hold \( \forall \) vectors \( a \). In particular, it must hold for the vector \( a \) with components

\[ a_1 = 1, \quad a_2 = a_3 = 0 \text{ w.r.t. } X. \]

With this choice of \( a \), (7.7) yields

\[ \lambda_j \lambda_{j1} = \delta_{i1} \]

Similar special choices of \( a \) lead to

\[ \lambda_j \lambda_{j2} = \delta_{i2} \text{ and } \lambda_j \lambda_{j3} = \delta_{i3}. \]

Thus
Similarly, starting again with the transformation rules but eliminating \( a_i \) instead of \( a'_i \).

Summing up, we have

**Theorem 7.2** The direction cosines satisfy the orthogonality condition

\[
\lambda_{ji} \lambda_{jk} = \lambda_{ij} \lambda_{kj} = \delta_{ik}.
\]

Matrices whose elements satisfy the orthogonality conditions are called orthogonal, and the transformation \( X \to X' (a_i \to a'_i) \) is said to be an orthogonal transformation.

An orthogonal matrix (and also an orthogonal transformation) is said to be proper or improper according as \( \det \lambda_{ij} = 1 \) or \(-1\).

Recalling that the elements of the transpose of \( [a_{ij}] \) are defined by

\[
a_{ij}^T = a_{ji},
\]

we can write the orthogonality conditions as

\[
\lambda_{ij}^T \lambda_{jk} = \lambda_{ij} \lambda_{jk}^T = \delta_{ik}.
\]

It makes it clear that in the language of Section 5, we can say that an orthogonal matrix is defined by the property that its transpose is its inverse. In view of Theorem 5.4, each of the orthogonality conditions implies the other.

8) **Cartesian Tensors**

As a generalization of the vector concept, we make the following definition.

**Definition 8.1** An entity represented in each rectangular Cartesian coordinate frame by \( 3^N \) real numbers

\[
a_{ij, \text{component in } X}
\]

\[
a'_{ij, \text{component in } X'}
\]

is called a Cartesian tensor of order \( N \) if under every orthogonal transformation \( X \to X' \), the components transform according to

\[
\lambda_{ji} \lambda_{jk} = \delta_{ik}.
\]
\[ a'_{ij...mn} = \lambda_{ij} \lambda_{iq} ... \lambda_{nr} a_{pq...rs} \]
\[ a_{ij...mn} = \lambda_{pi} \lambda_{qj} ... \lambda_{rm} a'_{pq...rs} \]  

(8.1) and (8.2)

Actually, (8.1) and (8.2) are equivalent.

Note that a vector is a Cartesian tensor of order one. A scalar is an entity represented by a single real number which is independent of coordinate systems and hence may be interpreted as a Cartesian tensor of order zero.

The most important tensor property is that if we know components in one frame, we can find them in any other by the transformation rule (8.1). In the physical sciences, one usually relies on particular coordinate frames to make measurements and to carry out theoretical derivations. The consistent use of tensors in the formulation of physical theories allows one to use coordinates and components whenever necessary or convenient without attaching special importance to any particular frame.

Operations with Cartesian tensors are defined so as to be consistent with indicial notation.

Definition 8.2 Let \( a_{ij...k} \) and \( b_{ij...k} \) be the components in \( X \) of two \( N^{th} \)-order Cartesian tensors.

(i) The two tensors are equal if their components agree in every frame; in particular
\[ a_{ij...k} = b_{ij...k} \cdot \]

(ii) The sum of the two tensors has the components
\[ a_{ij...k} + b_{ij...k} \] in \( X \),
similarly for other frames.
The scalar multiple of a tensor \( a_{ij...k} \) (components in \( X \)) by a scalar \( \lambda \) has the components

\[ \lambda a_{ij...k} \text{ in } X, \]
similarly for other frames.

**Theorem 8.1** The operations of addition and scalar multiplication yield Cartesian tensors of the same order.

**Theorem 8.2** If the components of a Cartesian tensor are all zero in one frame, then they are all zero in any other frame.

**Theorem 8.3** If the components of two tensors (of the same order) agree in any one frame, then the two tensors are equal.

**Theorem 8.4** The operation of interchange of subscripts on tensor components results in a tensor of the same order.

**Theorem 8.5** Let \( a_{ij...m...n...k} \) be the components in \( X \) of a Cartesian tensor. If \( a_{ij...m...n...k} \) is symmetric (skew) w.r.t. \( m \) and \( n \), then the components in any other frame are also symmetric (skew) w.r.t. \( m \) and \( n \).

**Theorem 8.6** Let \( a_{ij...k} \) and \( b_{pq...r} \) be components in \( X \) of Cartesian tensors.

Then the numbers defined in each frame \( X \) by the construction

\[ c_{ij...kpq...r} = a_{ij...k} b_{pq...r} \]

are the components (in \( X \)) of a Cartesian tensor of order \( M+N \).

The "c tensor" is called the outer product of the "a tensor" and the "b tensor".

**Theorem 8.7** Let \( a_{ij...m...n...k} \) be the components in \( X \) of a Cartesian tensor of order \( N \). Then the numbers defined in each frame \( X \) by the construction

\[ b_{ij...k} = a_{ij...m...n...k} \]

are the components (in \( X \)) of a Cartesian tensor of order \( N-2 \).

The "b tensor" is said to be formed from the "a tensor" by contraction.
If the operation of contraction is applied to the outer product of two tensors in such a way that the repeated index belongs to each of the two factors, the resulting tensor is called an **inner product**.

We have just seen, e.g., that if $a_{ij}$ and $b_{ij}$ are Cartesian tensor components, then $c = a_{ij}b_{ij}$ is a scalar. It is natural to ask: If $c$ and $b_{ij}$ are tensor components, are the $a_{ij}$ necessarily tensor components? Theorems that answer questions of this sort are called **quotient laws**. The following two theorems are typical quotient laws.

**Theorem 8.8** Suppose that in each frame we have a set of 9 real numbers, e.g., $a_{ij}$ in $X$.

Suppose that in each frame $X$

$$a_{ij}b_{ij} = c$$

for every choice of the vector $b_{j}$, where the $c_{i}$ are also vector components. Then the $a_{ij}$ are the components (in $X$) of a Cartesian tensor.

Note that if we have a set of $3^{N}$ real numbers associated with a single frame, say $X$, then we can construct a Cartesian tensor of order $N$ by defining components in any other frame via the transformation rule (8.1). We now construct tensors this way out of Kronecker's delta and the alternating symbol.

We start with Kronecker's delta and take the components in $X$ to be $\delta_{ij}$ (Definition 3.1). Then the components in any other frame $X$ are defined by

$$\delta'_{ij} = \lambda_{im} \delta_{mj} \delta_{mn}$$

$$= \lambda_{im} \delta_{jm}$$ (by Theorem 3.2)

$$= \delta_{ij}$$ (by Theorem 7.2)
Thus, we have

**Theorem 8.10** The entity represented in each rectangular Cartesian coordinate frame by Kronecker’s delta $\delta_{ij}$ is a second-order Cartesian tensor - the identity tensor.

In general, tensors whose components are the same in every frame are said to be **isotropic**.

Turning now to the alternator, we take the components in $X$ to be $\varepsilon_{ijk}$ (Definition 3.5). Then the components in any other frame $X'$ are defined by

$$
\varepsilon'_{ijk} = \lambda_{ip} \lambda_{jq} \lambda_{kr} \varepsilon_{pqr} = \det[\lambda_{mn}] \varepsilon_{ijk} \quad \text{(by Theorem 4.3)}
$$

$$
= +\varepsilon_{ijk} \quad \text{or} \quad -\varepsilon_{ijk} \quad \text{(by a homework exercise)}
$$

according as the orthogonal transformation $X \rightarrow X'$ is proper or improper.

This + or - property is the basis for the name alternator.

It is customary to define the alternator as having the components $\varepsilon_{ijk}$ in every rectangular Cartesian coordinate frame. Then the transformation law is

$$
\varepsilon'_{ijk} = \varepsilon_{ijk} = \det[\lambda_{mn}] \lambda_{ip} \lambda_{jq} \lambda_{kr} \varepsilon_{pqr}.
$$

Sets of numbers which transform according to this type of rule (weighted by the factor $\det[\lambda_{mn}]$) are known as **axial** or **pseudo** or **weighted** Cartesian tensors.

In order to avoid these difficulties, we will (unless explicitly stated to the contrary) restrict our definition of Cartesian tensors so as to admit only proper orthogonal transformations. Thus, we have

**Theorem 8.11** The entity represented in each rectangular Cartesian coordinate frame by the alternating symbol $\varepsilon_{ijk}$ is a third-order Cartesian tensor - the **alternating tensor**.
9) Scalar, Vector, and Tensor Products of Vectors

Definition 9.1 Let \( a \) and \( b \) be vectors (components \( a_i \) and \( b_i \) in \( X \)). Then the scalar (or dot or inner) product of \( a \) and \( b \) is

\[
 a \cdot b = a_i \cdot b_i .
\]

It follows immediately from Theorems 8.6 and 8.7 that \( a \cdot b \) is indeed a scalar (tensor of order zero); i.e., \( a \cdot b \) is independent of the choice of frame.

Theorem 9.1 The scalar product is commutative and distributive w.r.t. addition, i.e.,

\[
 a \cdot b = a \cdot b , \\
 a \cdot (b + c) = a \cdot b + a \cdot c , \\
 (\lambda \cdot a) \cdot b = \lambda \cdot (a \cdot b) .
\]

There is no question about the scalar product being associative since \( (a \cdot b) \cdot c \) has no meaning because \( a \cdot b \) is not a vector.

As a special case of the scalar product, we can write the length of \( a \) as

\[
 |a| = \sqrt{a \cdot a}
\]

The geometrical significance of the scalar product is contained in

Theorem 9.2 Let \( \theta \) be the angle between \( a \) and \( b \) \( (0 \leq \theta \leq \pi) \). Then

\[
 a \cdot b = |a| |b| \cos \theta
\]

As corollaries, we have

Theorem 9.3 The components of \( a \) are given by

\[
 a_i = e_i \cdot a
\]

Theorem 9.4 Suppose \( a, b \neq 0 \). Then

\[
 a \cdot b = 0
\]

iff \( a \) and \( b \) are orthogonal (i.e., the angle between them is \( \frac{\pi}{2} \)).
Definition 9.2 Let $a$ and $b$ be vectors (components $a_i$ and $b_i$ in $X$). Then the vector (or cross) product of $a$ and $b$ is the vector $c$ whose components in $X$ are
\[ c_i = \varepsilon_{ijk} a_j b_k. \]
We write $c = a \times b$. It follows immediately from Theorems 8.11, 8.6, and 8.7 that $a \times b$ is indeed a vector.

Theorem 9.5 The vector product is anti-commutative, distributive w.r.t. addition, and nonassociative; i.e.,
\[
\begin{align*}
    a \times b &= -b \times a, \\
    a \times (b + c) &= a \times b + a \times c, \\
    a \times (b \times c) &= (a \cdot c)b - (a \cdot b)c, \\
    (a \times b) \times c &= (a \cdot c)b - (b \cdot c)a, \\
    (\lambda a) \times b &= \lambda (a \times b).
\end{align*}
\]
In terms of the unit vectors $\{e_i\}$ for $X$, we can write
\[ a \times b = \varepsilon_{ijk} a_j b_k e_i. \]
This equation together with (Hmk. 1, Prob. 7) gives rise to the familiar mnemonic
\[ a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \]
The geometrical significance of the vector product is contained in

Theorem 9.6 Let $n$ be a unit vector perpendicular to the plane of $a$ and $b$ the ordered triple $a,b,n$ is right-handed or left-handed according as the frame $X$ used in the definition of $a \times b$ is right-handed or left-handed. Let $\theta$ ($0 \leq \theta \leq \pi$) be the angle between $a$ and $b$. Then
\[ a \times b = |a||b|\sin \theta n \]
The above theorem appears to indicate that the frame $X$ used in the definition of the cross product has some special significance. However, we shall soon see that if $X$ is right (left)-handed, then all frames that can be obtained from $X$
by proper orthogonal transformations are also right (left)-handed. We have already agreed to admit only proper orthogonal transformations and now we agree to work with only right-handed frames. Then the ordered triple \(a, b, a \times b\) is always a right-handed triple.

A frequently encountered expression is the scalar triple product

\[ a \cdot (b \times c). \]

It follows directly from Definitions 9.1 and 9.2 that

\[ a \cdot (b \times c) = \varepsilon_{ijk} a_i b_j c_k \]

and then from (Hmk.1, Prob. 7) that

\[ a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \]

The scalar triple product has an important geometrical interpretation. First consider the parallelogram determined by the vectors \(b\) and \(c\).

The area of this parallelogram is \(|b||c|\sin \theta = |b \times c|\). Now consider the parallelopiped determined by the three vectors \(a, b, c\).

The volume of this parallelopiped is \( |b \times c||d| \cos \phi \). Hence, the volume of the parallelopiped is \( +a \cdot (b \times c) \) or \( -a \cdot (b \times c) \) according as the ordered triple is right- or left-handed.

Now let \(X\) be a right-handed rectangular Cartesian coordinate frame, and let \(X'\) be any other rectangular Cartesian coordinate frame - not necessarily right-handed. The transformation \(X \to X'\) is characterized by the orthogonal matrix \(\lambda_{ij}\), which is not necessarily proper orthogonal. Consider the scalar triple product \(e'_i \cdot (e'_2 \times e'_3)\). Since

\[ e'_i = \lambda_{ij} e_j \]

\[ e'_i \cdot (e'_2 \times e'_3) = \det[\lambda_{ij}]. \]
The volume of the parallelopiped determined by $e'_1, e'_2, e'_3$ is 1, and from the discussion above $e'_1 \cdot (e'_2 \times e'_3) = 1$ or $-1$ according as the ordered triple $e'_1, e'_2, e'_3$ is right- or left-handed. Thus, we have

**Theorem 9.7** If the frame $X$ is right-handed, andy frame $X'$ related to $X$ by a proper (improper) orthogonal transformation is right (left) - handed.

We emphasize again that we will work only with proper orthogonal transformations and only with right-handed frames. In view of Theorem 9.7, we can do this consistently.

**Theorem 9.8** Suppose $a, b \neq 0$. Then

$$a \times b = 0$$

iff $a$ and $b$ are collinear (i.e., $a = kb$ for some nonzero scalar $k$).

**Definition 9.3** Let $a$ and $b$ be vectors (components $a_i$ and $b_i$ in $X$). Then the tensor (or dyadic or outer) product of $a$ and $b$ is a second-order Cartesian tensor whose components in $X$ are

$$a_i b_j .$$

It follows immediately from Theorem 8.6 that the tensor product is indeed a second-order Cartesian tensor. Symbolically, it is denoted by

$$a \otimes b \text{ or } ab .$$

The fact that a skew tensor has only 3 independent components suggests that it can be replaced by a vector, and we are lead to

**Theorem 9.9** Given any skew tensor $W_{ij}$, $\exists$ a unique vector $w_i \ni$

$$W_{ij}u_j = \epsilon_{ijk}w_ku_k = (w \times u)_i, \forall \text{ vectors } u$$

In fact,

$$w_j = -\frac{1}{2}\epsilon_{ijk}W_{jk}$$
$w$ is called the **axial vector** or **dual vector** corresponding to $W_{ij}$. Conversely, given any vector $w$, $\exists$ a unique skew tensor $W_{ij} \ni$

$$W_{ij}u_j = \epsilon_{ijk}w_k \forall \text{ vectors } u.$$  

In fact,

$$W_{ij}u_j = -\epsilon_{ijk}w_k.$$  

**Definition 9.4** Let $a$ and $b$ be vectors (components $a_i$ and $b_i$ in $X$). Then the skew (or exterior) product of $a$ and $b$ is the second-order Cartesian tensor whose components in $X$ are

$$a_ib_j - a_jb_i$$

In view of (Hmk. 1, Prob. 1), we can say that the skew product of $a$ and $b$ is twice the skew symmetric part of the tensor product $a \otimes b$. The skew product of $a$ and $b$ is usually denoted by

$$a \wedge b,$$

which unfortunately is also occasionally used for the cross product.

**Theorem 9.10** Given two vectors $a$ and $b$,

$$a \times b = -\text{dual}(a \wedge b),$$

where $\text{dual}(a \wedge b)$, denotes the dual vector of the skew product $a \wedge b$.

---

**10) Principal Directions of Symmetric Second-Order Tensors**

In this section, all assertions are restricted to symmetric second-order tensors.

**Definition 10.1** The unit vector $n$ (components $n_i$ in $X$) is said to be a principal direction (or **eigenvector** or **characteristic vector**) of the symmetric second-order Cartesian tensor $a_{ij}$ (components in $X$) if $\exists$ a real number $a \ni$

$$a_{ij}n_j = an_i$$

i.e., if the vector $a_{ij}n_j\epsilon_i$ is parallel to $n$. The number $a$ is called the associated **principal value** (or **eigenvalue** or **characteristic number**) of the tensor.
Theorem 10.1  The base vector $e_1$ is a principal direction of $a_{ij}$ (components in X) iff

$$
[a_{ij}] = \begin{bmatrix}
a_{11} & 0 & 0 \\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{bmatrix},
$$

and in this case $a_{11}$ is the associated principal value. Similarly, for the other base vectors.

Theorem 10.2  Principal directions associated with distinct principal values are orthogonal.

Now suppose that the unit vector $n$ is a principal direction of $a_{ij}$ with associated principal value $a$ so that

$$
a_{ij}n_j = an_i.
$$

These equations are easier to study if we note that $n_i = \delta_{ij}n_j$ and write them in the form

$$
(a_{ij} - a\delta_{ij})n_j = 0.
$$

This may be viewed as a system of 3 homogeneous linear, algebraic equations in the 3 $n_j$'s, and by Theorem 5.7, $\exists$ nontrivial solutions (required by $|n| = 1$) if

$$
\det[a_{ij} - a\delta_{ij}] = 0
$$

Of course, any nontrivial solution can be normalized so that $n \cdot n = 1$.

Thus, we have

Theorem 10.3  Let $n$ be a principal direction of $a_{ij}$ (components in X) with associated principal value $a$. Then $a$ meets

$$
\det[a_{ij} - a\delta_{ij}] = 0.
$$

Conversely, suppose that $a \in R$ meets the above equation. Then $a_{ij}$ has a principal direction with associated principal value $a$. The unit vector $n$ defining such a direction is found by solving

$$
(a_{ij} - a\delta_{ij})n_j = 0
$$
subject to the condition \( n_j n_j = 1 \).

Our major result is

**Theorem 10.4** Every (symmetric) tensor \( a_{ij} \) (components in \( X \)) has at least one principal frame \( \hat{X} \); i.e., a right-handed triplet of orthogonal principal directions, and at most 3 distinct principal values. The principal values are the roots of the characteristic polynomial

\[
\det[a_{ij} - a\delta_{ij}] = -a^3 + I_1a^2 - I_2a + I_3 = 0,
\]

where the coefficients

\[
I_1 = a_{ii},
\]

\[
I_2 = \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ji}),
\]

\[
I_3 = \det[a_{ij}]
\]

are the fundamental invariants. The characteristic polynomial always has 3 roots, which are denoted by \( a_i \). The components in \( X \) of the principal axis \( \hat{e}_k \) corresponding to the principal value \( a_k \) are found by solving

\[
(a_{ij} - a\delta_{ij})\begin{pmatrix} \hat{e}_k \\ j \end{pmatrix} = 0
\]

subject to \( \hat{e}_k \cdot \hat{e}_k = 1 \). \( \hat{e}_k \) is not unique. Three possibilities arise.

**Case 1** (Principal Values Distinct) The corresponding principal axes are unique except for sense. Hence, \( \exists \) essentially only one principal frame.

**Case 2** (Two Principal Values Equal) Suppose that \( a_1 \neq a_2 = a_3 \) and that

\[
\hat{X} = \left\{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \right\}
\]

is a corresponding principal frame. Then \( \hat{e}_1 \) is unique to within sense, and every frame obtained by a rotation of \( \hat{X} \) about \( \hat{e}_1 \) is also a principal frame.

**Case 3** (All Three Principal Values Equal) Every right-handed frame is a principal frame.

**Proof:** The expression of the characteristic polynomial in terms of the fundamental invariants follows at once from expanding the determinant.
\[
\det[a_{ij} - a\delta_{ij}]
\]

according to (Hmk.1, Prob. 9). The details are left as a homework problem.

Since the characteristic polynomial is a cubic, it has 3 roots by the fundamental theorem of algebra. Since the coefficients are real, any complex roots occur in conjugate pairs. Thus, \( \exists \) at least on real root, say \( a_1 \). By Theorem 10.3, \( a_1 \) is a principal value of \( a_{ij} \), and the corresponding principal direction \( n_1 \) is found by solving
\[
(a_{ij} - a_1\delta_{ij})(n_i)_j = 0, \text{ subject to } n_1 \cdot n_1 = 1
\]

Let us choose \( X' \ni e'_1 = n_1 \). By Theorem 10.1,
\[
\begin{bmatrix}
    a_{11} & 0 & 0 \\
    0 & a_{22}' & a_{23}' \\
    0 & a_{32}' & a_{33}'
\end{bmatrix}
\]

By (Hmk.1, Prob.9) and Theorems of Section 8, the characteristic polynomial is a tensor of order zero; i.e., an invariant w.r.t. rotations of frames.

Accordingly, we look at it in \( X' \) to get
\[
\det[a_{ij} - a\delta_{ij}] = \det[a'_{ij} - a\delta_{ij}]
\]

\[
= \begin{vmatrix}
    (a_i - a) & 0 & 0 \\
    0 & (a_{22}' - a) & a_{23}' \\
    0 & a_{32}' & (a_{33}' - a)
\end{vmatrix}
\]

\[
= (a_i - a)[a^2 - (a_{22}' + a_{33}')a + a_{22}'a_{33}' - a_{23}']
\]

By the quadratic formula, this cubic has the 3 real roots:
\[
a_1
\]
\[
a_2, a_3 = \frac{1}{2}(a_{22}' + a_{33}') \pm \sqrt{(a_{22}' - a_{33}')^2 + (2a_{23}')^2}
\]

Again by Theorem 10.3, the corresponding principal directions \( n_1, n_2, n_3 \) are found by solving
\[
(a_{ij} - a_k\delta_{ij})(n_k)_j = 0, \text{ subject to } n_k \cdot n_k = 1.
\]
Case 1 \((a_1 \neq a_2 \neq a_3 \neq a_1)\) By Theorem 10.2; \(n_1, n_2,\) and \(n_3\) are mutually perpendicular. Hence \(\hat{X} = \{\hat{e_1}, \hat{e_2}, \hat{e_3}\}\) defined by \(\hat{e_i} = \pm n_i\) is a principal frame. 

(We can always choose the signs \(\in X\) is right-handed).

Now suppose that \(\hat{X} = \{\hat{e_1}, \hat{e_2}, \hat{e_3}\}\) is any other principal frame. To show that \(\hat{X}\) is not essentially different from \(\hat{X}\), we note from Theorem 10.2 that 
\[
\hat{e}_k \cdot \hat{e}_l = 0 \quad \text{for} \quad k \neq l
\]

Thus, e.g., \(\hat{e}_1\) is perpendicular to both \(\hat{e}_2\) and \(\hat{e}_3\) \(\Rightarrow\) \(\hat{e}_1 = \pm \hat{e}_1^*\).

Similarly, \(\hat{e}_2 = \pm \hat{e}_2^*, \hat{e}_3 = \pm \hat{e}_3^*\).

Case 2 \((a_1 \neq a_2 = a_3)\) As in our analysis of the characteristic polynomial, we make the change of frame \(X \to X' \ni \hat{e}_1^* = n_1\). Then from (10.1),
\[
\begin{align*}
a_2 &= \frac{1}{2} (a_{22}' + a_{33}') + \sqrt{(a_{22}' - a_{33}')^2 + (2a_{23}')^2}, \\
a_3 &= \frac{1}{2} (a_{22}' + a_{33}') - \sqrt{(a_{22}' - a_{33}')^2 + (2a_{23}')^2};
\end{align*}
\]

and \(a_2 = a_3 \Rightarrow a_{22}' = a_{33}', a_{23}' = 0\) and \(a_{22}' = a_{33}' = a_2 = a_3\).

Thus,
\[
[a_{ij}] = \begin{bmatrix} a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3 \end{bmatrix},
\]

and by Theorem 10.1, \(X'\) is a principal frame. \(e_i^*\) is unique to within sense by the argument used in Case 1. However, there are no constraints on the choice of \(e_2'\) and \(e_3'\) other than orthogonality and right-handedness.

Case 3 \((a_1 = a_2 = a_3 = a)\) As in Case 1 and 2, we make the change of frame \(X \to X' \ni e_1^* = n_1\). Then
\[
[a_{ij}] = \begin{bmatrix} a & 0 & 0 \\
0 & a & 0 \\
0 & 0 & a \end{bmatrix} \Rightarrow X'\text{ a principal frame.}
\]

Since
\[ a'_{ij} = a \delta_{ij}, \]
\[ a'_i n'_j = a \delta_{ij} n'_j = a n'_i; \]
i.e., every unit vector \( n \) defines a principal axis. Consequently, all frames are principal. ("end of proof")

The principal values of a symmetric tensor have important extremal properties which we introduce in the following geometrical way. Let \( a_{ij} \) (components in \( X \)) be a symmetric but otherwise arbitrary second-order Cartesian tensor. Then for any unit vector \( n \)

\[ a(n) = a_{ij} n^j e^i = a_{ij} n^j e^i = a' n_i; \]

is a vector. The component of \( a(n) \) in the \( n \) direction is the quadratic form

\[ N(n) = n \cdot a(n) = n_i a_{ij} n^j = a_{ij} n^i n^j. \]

We can express \( a(n) \) as

\[ a(n) = N(n)n + s(n) \]

where

\[ s(n) \cdot n = 0. \]

Thus, \( s(n) \) is the vector component of \( a(n) \) on the plane normal to \( n \). We define

\[ S(n) = |s(n)|, \]

and note that

\[ |a(n)|^2 = |N(n)|^2 + |S(n)|^2. \]

Now take \( \hat{X} \) to be a principal frame for \( a_{ij} \) so that

\[
\begin{bmatrix}
  a_1 & 0 & 0 \\
  0 & a_2 & 0 \\
  0 & 0 & a_3 \\
\end{bmatrix},
\]

and then (10.2) becomes

\[ a(n) = a_{ij} n^i e^i + a_{ij} n^i e^i + a_{ij} n^i e^i \]

Consequently,
\[
N^2 + S^2 = a(n) = a_1^2 n_1^* + a_2^2 n_2^* + a_3^2 n_3^*,
\]
\[
N = a_1 n_1^* + a_2 n_2^* + a_3 n_3^*,
\]
\[
1 = n_1^* + n_2^* + n_3^*.
\]
\[
(10.3)
\]

Equations (10.3) may be viewed as a set of 3 linear, algebraic equations for the \(3 n_i^*\). Since \(n_i^* \geq 0\), these equations place restrictions in \(N\) and \(S\). In order to investigate these, we assume that the principal values have been ordered:

\[
a_1 \geq a_2 \geq a_3
\]

As in Theorem 10.4, there are three possibilities.

**Case 1** (Principal Values Distinct, \(a_1 > a_2 > a_3\)) In this case, the determinant of the coefficient matrix is

\[
(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) \neq 0
\]

and the solution of (10.3) is

\[
0 \leq n_1^* = \frac{S^2 + (N - a_2)(N - a_3)}{(a_1 - a_2)(a_1 - a_3)},
\]

\[
0 \leq n_2^* = \frac{S^2 + (N - a_3)(N - a_1)}{(a_2 - a_3)(a_2 - a_1)},
\]

\[
0 \leq n_3^* = \frac{S^2 + (N - a_1)(N - a_2)}{(a_3 - a_1)(a_3 - a_2)}.
\]

Since \(a_1 > a_2 > a_3\), these inequalities \(\Rightarrow\)

\[
S^2 + (N - a_2)(N - a_3) \geq 0,
\]

\[
S^2 + (N - a_3)(N - a_1) \leq 0,
\]

\[
S^2 + (N - a_1)(N - a_2) \geq 0.
\]

Completing the square, we rewrite this last set of inequalities as

\[
S^2 + \left[ N - \left( \frac{a_2 + a_3}{2} \right) \right]^2 \geq \left( \frac{a_2 - a_3}{2} \right)^2,
\]

\[
S^2 + \left[ N - \left( \frac{a_3 + a_1}{2} \right) \right]^2 \leq \left( \frac{a_3 - a_1}{2} \right)^2,
\]

\[
S^2 + \left[ N - \left( \frac{a_1 + a_2}{2} \right) \right]^2 \geq \left( \frac{a_1 - a_2}{2} \right)^2.
\]

\[
(10.4)
\]
Inequalities (10.4) are easily interpreted by letting N and S represent the abscissa and ordinate, respectively, of a point in a plane. Then any admissible point (N,S) must lie in the cross-hatched region between the 3 semicircles as shown below. (Recall that $S \geq 0$)

**Case 2** (Two Principal Values Equal, say $a_1 > a_2 = a_3$) In this case, (10.3) reduce to

\[
\begin{align*}
N^2 + S^2 &= a_1^2 n_1^2 + a_2^2 \left( n_2^2 + n_3^2 \right) \\
N &= a_1 n_1^2 + a_2 \left( n_2^2 + n_3^2 \right) \\
1 &= n_1^2 + \left( n_2^2 + n_3^2 \right)
\end{align*}
\]

Substitution for $\left( n_2^2 + n_3^2 \right)$ from the third of these into the first two yields

\[
(a_1^2 - a_2^2)n_1^2 + a_2^2 = N^2 + S^2
\]

\[
n_1^2 = \frac{N - a_2}{a_1 - a_2}
\]

Eliminating $n_1^2$, we find that

\[
S^2 + \left[ N - \left( a_1 + a_2 \right) \right]^2 = \left( \frac{a_1 - a_2}{2} \right)^2.
\]

Thus, in Case 2, the point (N,S) must lie on the semicircle indicated below.

We note that this semicircle is what results when the admissible region of Case 1 collapses in accordance with $a_3 = a_2$. The inequalities (10.4) still hold in Case 2.

**Case 3** (All Three Principal Values Equal, $a_1 = a_2 = a_3$) In this case, (10.3) reduces to

\[
N = a_1, \quad S = 0
\]

i.e., the set of admissible (N,S) points is the single point $(a_1,0)$. 


Of course, this agrees with the collapsing of the admissible region of Case 1 in accordance with \( a_1 = a_2 = a_3 \). The inequalities (10.4) obviously still hold in Case 3.

It is clear from the above figures that in any case,
\[
\max_{\|\mathbf{n}\| = 1} N(n) = a_1,  \\
\min_{\|\mathbf{n}\| = 1} N(n) = a_3,  \\
\max_{\|\mathbf{n}\| = 1} S(n) = \frac{a_1 - a_3}{2}.
\]

To find the values of \( n \) at which these extrema occur, we merely substitute the corresponding values of \( N \) and \( S \) into (10.3). E.g., to find the \( n \) at which \( \max N \) occurs we substitute \( N = a_1, S = 0 \) into (10.3) to get
\[
\begin{align*}
    a_1^2 & = a_1^* n_1^2 + a_2^* n_2^2 + a_3^* n_3^2,  \\
    a_1 & = a_1^* n_1^2 + a_2^* n_2^2 + a_3^* n_3^2,  \\
    1 & = n_1^2 + n_2^2 + n_3^2.
\end{align*}
\]

These equations may be solved by considering the 3 possibilities of relative sizes of the principal values just as before. There will, of course, be some nonuniqueness in Cases 2 and 3. In any case, the result is that \( \max N \) occurs at \( n_1^* = \pm 1, n_2^* = n_3^* = 0 \). Of course, the star refers to the fact that the components are w.r.t. the principal frame \( \mathbf{X}^* \).

In the same way, we find that \( \min N \) occurs at \( n_1^* = n_2^* = 0, n_3^* = \pm 1 \) and that \( \max S \) occurs at \( n_1^* = \pm \frac{1}{\sqrt{2}}, n_2^* = 0, n_3^* = \pm \frac{1}{\sqrt{2}} \).

These results are summed up in the following theorem. The geometrical construction that we used in the development is known as 3-dimensional Mohr's circles.

**Theorem 10.5** Let \( a_{ij} \) (components in \( \mathbf{X} \)) be a symmetric Cartesian tensor. For arbitrary unit vector \( \mathbf{n} \), define the vector
\[
a(n) = a_{ij} n_j \mathbf{e}_i
\]
and write

\[ a(n) = N(n)n + s(n), \quad n \cdot s(n) = 0. \]

Set

\[ S(n) = |s(n)|. \]

Order the principal values of \( a_{ij} \) according to

\[ a_1 \geq a_2 \geq a_3. \]

Then

\[
\begin{align*}
\max_{|n|=1} N(n) &= a_1, \\
\min_{|n|=1} N(n) &= a_3, \\
\max_{|n|=1} S(n) &= \frac{a_1 - a_3}{2}.
\end{align*}
\]

Moreover, the direction \( n \) at which \( \max N \) (\( \min N \)) occurs is a principal direction of \( a_{ij} \) corresponding to the largest (smallest) principal value \( a_1 \) (\( a_3 \)), while the direction of \( \max S \) bisects the directions of \( \max N \) and \( \min N \).

11) Cartesian Tensor Fields

In general in mechanics, functions of positions are called **fields**. Let \( P \) be a typical point of \( \varepsilon \) and let \( X \) be an arbitrary rectangular Cartesian coordinate frame.

Of course, given \( X \), \( P \) is determined by its coordinates \( x_i \) or its position vector \( \rightarrow OP = x \). Moreover, \( x_i = e_i \cdot x \). Thus, a function of position, say \( f \), can be thought of as a function of the coordinates \( x_i \) or the position vector \( x \), and we write

\[ f(P) = f(x_1, x_2, x_3) = f(x_i) = f(x) \]

as best suits the purpose at hand. Of course, the actual functions above cannot really be the same because their domains are different; however, their function values agree at corresponding arguments.
Generally, there is nothing special about the frame $X$, and in the spirit of Cartesian tensor analysis, we consider a second rectangular frame $X'$. We keep the origin of $X'$ at 0 so that the position vector $P$ w.r.t. $X'$ is still $\overrightarrow{OP} = x$. The coordinates of $P$ w.r.t. $X'$ are just the components of $\overrightarrow{OP} = x$ w.r.t. $X'$, i.e.,

$$x'_i = e'_i \cdot x = \lambda_{ij} x_j$$

**Definition 11.1** If at each point $P$ in some subset of $\varepsilon$

$$a_{ij...k}(P) \quad \text{(components in $X$)}$$

are components of a Cartesian tensor of order $N$, then they are said to be the components of a Cartesian tensor field of order $N$. Writing

$$a'_{ij...k}(P)$$

for the components in $X'$, we have the transformation rule

$$a'_{ij...k}(P) = \lambda_{im} \lambda_{jm} \ldots \lambda_{kp} a_{mn...p}(P).$$

Note that the $\lambda$'s are not functions of position. It is usually most convenient to view components w.r.t. $X$ as functions of the coordinates of $P$ w.r.t. $X$, etc.

For the remainder of this section, $D$ will always denote some open subset of $\varepsilon$.

**Definition 11.2** Let $f$ be a real-valued function on $D$. Then we say that $f$ is in class $C^M$ on $D$ and write

$$f \in C^M(D)$$

if $f$ and all its partial derivatives up to and including order $M$ are continuous on $D$.

Of course, the partial derivatives above refer to viewing $f$ as a function of the coordinates (w.r.t. some frame) of the points in $D$. The kind of analysis used in proving the following theorem can be used to show that the choice of frame here is not significant.

**Theorem 11.1** Let $a_{ij...k}(P) \in C'(D)$ be the components in $X$ of a Cartesian tensor field of order $N$. Define
\[ b_{ij...k}(x_1, x_2, x_3) = \frac{\partial}{\partial x_i} a_{ij...k}(x_1, x_2, x_3) \quad \text{(components in } X) \]

\[ b'_{ij...k}(x'_1, x'_2, x'_3) = \frac{\partial}{\partial x'_i} a'_{ij...k}(x'_1, x'_2, x'_3) \quad \text{(components in } X') \]

Then the \( b' \)'s are the components of a Cartesian tensor field of order \( N+1 \).

**Definition 11.3 (Comma Notation)**

\[ a_{ij...k,m} = \frac{\partial}{\partial x_m} a_{ij...k} \, , \]

\[ a'_{ij...k,m} = \frac{\partial}{\partial x'_m} a'_{ij...k} \, . \]

**Definition 11.4** Given a scalar field \( \phi \) and a vector field \( v \) of sufficient smoothness, we define

- gradient of \( \phi \) : \( \text{grad} \phi = \nabla \phi = \phi_i e_i \)
- Laplacian of \( \phi \) : \( \Delta \phi = \nabla^2 \phi = \phi_{ii} \)
- divergence of \( v \) : \( \text{div} \, v = \nabla \cdot v = v_{i,j} \)
- curl of \( v \) : \( \text{curl} \, v = \nabla \times v = \varepsilon_{ijk} v_j e_i \)
- Laplacian of \( v \) : \( \Delta v = \nabla^2 v = (\Delta v_i) e_i = v_{i,j} e_i \)

**Theorem 11.2** Let \( \theta \) and \( \phi \) be scalar fields and \( u \) and \( v \) be vector fields, all of class \( C'(D) \). Then we have the following identities

1. \( \nabla(\theta \phi) = \phi(\nabla \theta) + \theta(\nabla \phi) \)
2. \( \nabla \cdot (\theta \, u) = \nabla \theta \cdot u + \theta \nabla \cdot u \)
3. \( \nabla \times (\theta \, u) = \nabla \theta \times u + \theta \nabla \times u \)
4. \( \nabla \cdot (u \times v) = v \cdot \nabla \times u - u \cdot \nabla \times v \)

**Theorem 11.3** Let \( \theta \) and \( \phi \) be scalar fields and \( u \) be a vector field, all of class \( C^2(D) \). Then

1. \( \Delta(\theta \, \phi) = (\Delta \theta) \phi + \theta(\Delta \phi) + 2 \nabla \theta \cdot \nabla \phi \)
2. \( \nabla \cdot (\nabla \theta) = \nabla \theta \)
3. \( \nabla \times (\nabla \theta) = 0 \)
4. \( \nabla \cdot (\nabla \times u) = 0 \)
5. \( \nabla \times (\nabla \times u) = \nabla \nabla \cdot u - \Delta u \).