Inventory Theory

Single Period Stochastic Inventories

This section considers an inventory situation in which the current order for the replenishment of inventory can be evaluated independently of future decisions. Such cases occur when inventory cannot be added later (spares for a space trip, stocks for the Christmas season), or when inventory spoils or becomes obsolete (fresh fruit, current newspapers). The problem may have multiple periods, but the current inventory decision must be independent of future periods. First we assume there is no setup cost for placing a replenishment order, and then we assume that there is a setup cost.

Single Period Model with No Setup Cost

Consider an inventory situation where the merchant must purchase a quantity of items that is offered for sale during a single interval of time. The items are purchased for a cost $c$ per unit and sold for a price $b$ per unit. If an item remains unsold at the end of the period, it has a salvage value of $a$. If the demand is not satisfied during the interval, there is a cost of $d$ per unit of shortage. The demand during the period is a random variable $x$ with given p.d.f. and c.d.f.. The problem is to determine the number of items to purchase. We call this the order level, $S$, because the purchase brings the inventory to level $S$. For this section, there is no cost for placing the order for the items.

The expression for the profit during the interval depends on whether the demand falls above or below $S$. If the demand is less than $S$, revenue is obtained only for the number sold, $x$, while the quantity purchased is $S$. Salvage is obtained for the unsold amount $S - x$. The profit in this case is

$$\text{Profit} = bx - cS + a(S - x) \text{ for } x \leq S.$$  

If the demand is greater than $S$, revenue is obtained only for the number sold, $S$. A shortage cost of $d$ is expended for each item short, $x - S$. The profit in this case is

$$\text{Profit} = bS - cS - d(x - S) \text{ for } x \geq S.$$  

Assuming a continuous distribution and taking the expectation over all values of the random variable, the expected profit is

$$E\{\text{Profit}\} = b\int_0^S x f(x) dx + b\int_S^\infty S f(x) dx - cS + a\int_0^S (S - x) f(x) dx - d\int_S^\infty (x - S) f(x) dx.$$
Rearranging and simplifying,

\[ E\{\text{Profit}\} = b\mu - cS + a\int_{0}^{\infty}(S-x)f(x)dx - (d + b)\int_{S}^{\infty}(x-S)f(x)dx. \]

We recognize in this expression the expected excess, \( E_e \), and the expected shortage, \( E_s \). The profit is written in these terms as

\[ E\{\text{Profit}\} = b\mu - cS + aE_e - (d + b)E_s \quad (27) \]

To find the optimum order level, we set the derivative of profit with respect to \( S \) equal to zero.

\[ \frac{dE\{\text{Profit}\}}{dS} = -c + a\int_{0}^{S}f(x)dx + (d + b)\int_{S}^{\infty}f(x)dx = 0. \]

or \(-c + aF(S) + (d + b)[1 - F(S)] = 0.\)

The c.d.f. of the optimum order level, \( S^* \), is determined by

\[ F(S^*) = \frac{b - c + d}{b - a + d}. \quad (28) \]

This result is sometimes expressed in terms of the purchasing cost, \( c \), a holding cost \( h \), expended for every unit held at the end of the period, and a cost \( p \), expended for every unit of shortage at the end of the period. In these terms the optimum expected cost is

\[ E\{\text{Cost}\} = cS + hE_e + pE_s. \]

The optimum solution has

\[ F(S^*) = \frac{p - c}{p + h}. \quad (29) \]

The two solutions are equivalent if we identify

\[ h = -a = \text{negative of the salvage value} \]
\[ p = b + d = \text{lost revenue per unit + shortage cost}. \]

If the demand during the period has a Normal distribution with mean \( \mu \) and standard deviation \( \sigma \), the expected profit is easily evaluated for any given order level. The order level is expressed in terms of the number of standard deviations from the mean, or

\[ S = \mu + k\sigma. \]

The optimality condition becomes

\[ \Phi(k^*) = \frac{b - c + d}{b - a + d} = \frac{p - c}{p + h}. \quad (30) \]
The expected value of profit is evaluated with the expression

\[ E(\text{Profit}) = b\mu - cS + a[S - \mu + \sigma G(k)] - (d + b)\sigma G(k). \]  

(31)

Call the quantity on the right of the Eq. 28 or 29 the threshold.

Optimality conditions for the order level give values for the c.d.f.. For continuous random variables there is a solution if the threshold is in the range from 0 to 1. No reasonable values of the parameters will result in a threshold less than 0 or larger than 1.

For discrete distributions the optimum value of the order level is the smallest value of \( S \) such that

\[ E(\text{Profit}|S + 1) \leq E(\text{Profit}|S + 1). \]

By manipulation of the summation terms that define the expected profit, we can show that the optimum order level is the smallest value of \( S \) whose c.d.f. equals or exceeds the threshold. That is

\[ F(S^*) \geq \frac{b - c + d}{b - a + d} \quad \text{or} \quad \frac{p - c}{p + h}. \]

(32)

**Example 4: Newsboy Problem**

The classic illustration of this problem involves a newsboy who must purchase a quantity of newspapers for the day's sale. The purchase cost of the papers is $0.10 and they are sold to customers for a price of $0.25. Papers unsold at the end of the day are returned to the publisher for $0.02. The boy does not like to disappoint his customers (who might turn elsewhere for supply), so he estimates a "good will" cost of $0.15 for each customer who is not be satisfied if the supply of papers runs out. The boy has kept a record of sales and shortages, and estimates that the mean demand during the day is 250 and the standard deviation is 50. A Normal distribution is assumed. How many papers should he purchase?

This is a single-period problem because today's newspapers will be obsolete tomorrow. The factors required by the analysis are

- \( a = 0.02 \), the salvage value of a newspaper,
- \( b = 0.25 \), the selling price of each paper,
- \( c = 0.10 \), the purchase cost of each paper,
- \( d = 0.15 \), the penalty cost for a shortage.

Since the demand distribution is Normal, we have from Eq. 30,

\[ \Phi(k^*) = \frac{b - c + d}{b - a + d} = \frac{0.25 - 0.10 + 0.15}{0.25 - 0.02 + 0.15} = 0.7895. \]

From the Normal distribution table, we find that

\[ \Phi(0.80) = 0.7881 \quad \text{and} \quad \Phi(0.85) = 0.8022. \]
With linear interpolation, we determine \( k^* = 0.805 \).

Then \( S^* = (0.805)(50) + 250 = 290.2 \).

Rounding up, we suggest that the newsboy should purchase 291 papers for the day. The risk of a shortage during the day is

\[ 1 - F(S^*) = 0.211. \]

Interpolating in the \( G \) Table 1 we find that

\[ G(k^*) = G(0.805) = 0.1192. \]

Then from Eq. 25, 26 and 31,

\[ E_e = 46.2, \ E_s = 5.96, \] and \( E\{Profit\} = $32.02 \) per day.

**Example 5: Spares Provisioning**

A submarine has a very critical component that has a reliability problem. The submarine is beginning a 1-year cruise, and the supply officer must determine how many spares of the component to stock. Analysis shows that the time between failures of the component is 6 months. A failed component cannot be repaired but must be replaced from the spares stock. Only the component actually in operation may fail; components in the spares stock do not fail. If the stock is exhausted, every additional failure requires an expensive resupply operation with a cost of $75,000 per component. The component has a unit cost of $10,000 if stocked at the beginning of the cruise. Component spares also use up space and other scarce resources. To reflect these factors a cost of $25,000 is added for every component remaining unused at the end of the trip. There is essentially no value to spares remaining at the end of the trip because of technical obsolescence.

This is a single-period problem because the decision is made only for the current trip. Failures occur at random, with an average rate of 2 per year. Thus the expected number of failures during the cruise is 2. The number of failures has a Poisson distribution. The second form of the solution, Eq. 29, is convenient in this case.

\[ h = 25,000, \] the extra cost of storage.

\[ c = 10,000, \] the purchase cost of each component.

\[ p = 75,000, \] the cost of resupply.

Expressed in thousands the threshold is

\[ F(S^*) = \frac{p - c}{p + h} = \frac{75 - 10}{75 + 25} = 0.65. \]

From the cumulative Poisson distribution using a mean of 2, we find

\[ F(0) = 0.135, \ F(1) = 0.406, \ F(2) = 0.677, \ F(3) = 0.857. \]

Since this is a discrete distribution, we select the smallest value of \( S \) such that the c.d.f. exceeds 0.65. This occurs for \( S^* = 2 \). We have the perhaps surprising result
that only two spares should be brought. This is in addition to the component initially installed, so that only on the third failure will a resupply be required. The probability of one or more resupply operations is
\[ 1 - F(2) = 0.323. \]
The resupply operation is an aspect of the situation that makes this model appropriate. If the system simply stopped after the spares were exhausted and a single cost of failure was expended, then the assumption of the linear cost of lost sales would be violated.

**Single Period Model with a Fixed Ordering Cost**

When the merchant has an initial source of product and there is a fixed cost for ordering new items, it may be less expensive to purchase no additional items than to order up to some order level. In this section we assume that initially there are \( z \) items in stock. If more items are purchased to increase the stock to a level \( S \), a fixed ordering charge, \( K \), is expended. We want to determine a level \( s \), called the reorder point, such that if \( z \) is greater than \( s \) we do not purchase additional items. Such a policy is called the reorder point, order level system, or the \((s, S)\) system.

We consider first the case where additional product is ordered to bring the inventory to \( S \) at the start of the period. The expression for the expected profit is the same as developed previously, except we must subtract the ordering charge and it is only necessary to purchase \((S - z)\) items.

\[
P_O(z, S) = b\mu - c(S - z) + aE_e(S) - (d + b)E_s(S) - K. \tag{33}
\]

We include the argument \( S \) with \( E_e(S) \) and \( E_s(S) \) to indicate that these expected values are computed with the starting inventory level at \( S \).

Neither \( z \) nor \( K \) affect the optimum solution, and as before

\[
F(S^*) = \frac{b - c + d}{b - a + d}
\]

If no addition items are purchased, the system must suffice with the initial inventory \( z \). The expected profit in this case is

\[
P_N(z) = b\mu + aE_e(z) - (d + b)E_s(z), \tag{34}
\]

where the expected excess and shortage depend on \( z \).

When \( z \) equals \( S \), \( P_N \) is greater than \( P_O \) by the amount \( K \), and certainly no additional items should be purchased. As \( z \) decreases, \( P_N \) and \( P_O \) become closer. The two expressions are equal when \( z \) equals \( s \), the optimum reorder point. Then the optimum reorder point is \( s^* \) where

\[
P_O(s^*, S) = P_N(s^*)
\]

Generally it is difficult to evaluate the integrals that allow this equation to be solved. When the demand has a Normal distribution, however, the
expected profit in the two cases can be written as a function of the distribution parameters.

Assuming a Normal distribution and given the initial supply, $z$, the profit when we replenish the inventory up to the level $S$ is

$$P_O(z, S) = b\mu - c(S - z) + a[S - \mu + \sigma G(k)] - (d + b)[\sigma G(k)] - K$$

(35)

Here $S = \mu + k\sigma$.

If we choose not to replenish the inventory, but rather operate with the items on hand the profit is

$$P_N(z) = E\{Profit\} = b\mu + a[z - \mu + \sigma G(k)] - (d + b)[\sigma G(k)]$$. (31)

Here $z = \mu + k\sigma$.

We modify the newsboy problem by assuming that the boy gets a free stock of papers each morning. The question is whether he should order more? The cost of placing an order is $10. In Fig. 8, we have plotted these the costs with and without an order. The profit is low when the initial stock is low and we do not reorder. The two curves cross at about 210. This is the reorder point for the newsboy. If he has 210 papers or less, he should order enough papers to bring his stock to 291. If he has more than 210 papers, he should not restock. The profit for a given day depends on how many papers the boy starts with. The higher of the two curves in Fig. 8 shows the daily profit if one follows the optimum policy. As expected the profit grows with the number of free papers.
Example 6: Demand with a Uniform Distribution

The demand for the next period is a random variable with a uniform distribution ranging from 50 to 250 units. The purchase cost of an item is $100. The selling price is $150. Items unsold at the end of the period go "on sale" for $20. All remaining are disposed of at this price. If the inventory is not sufficient, sales are lost, with a penalty equal to the selling price of the item. The current level of inventory is 100 units. Additional items may be ordered at this time; however, a delivery fee will consist of a fixed charge of $500 plus $10 per item ordered. Should an order be placed, and if so, how many items should be ordered?

To analyze this problem first determine the parameters of the model.

\[ c = \$110, \text{ the purchase cost plus the variable portion of the delivery fee} \]
\[ K = \$500, \text{ the fixed portion of the delivery fee} \]
\[ p = \$150, \text{ the lost income associated with a lost sale} \]
\[ h = -\$20, \text{ the negative of the salvage value of the product.} \]

From Eq. 29, the order level is \( S \), such that

\[ F(S^*) = \frac{p - c}{p + h} = \frac{150 - 110}{150 - 20} = 0.3077. \]

Setting the c.d.f. for the Uniform distribution equal to this value and solving for \( S \),
\[
F(S) = \frac{S - 50}{250 - 50} = 0.3077 \text{ or } S = 111.5.
\]

Rounding up, we select \( S^* = 112 \).

Modifying the expected cost function to include the initial stock and the cost of placing and order.

\[
C_O = c(S - z) + hE_e(S) + pE_s(S) + K.
\]

For the uniform distribution ranging from \( A \) to \( B \),

\[
E_e(S) = \frac{1}{(B - A)} \int_A^S (S - x) \, dx = \frac{(S - A)^2}{2(B - A)}.
\]

\[
E_s(S) = \frac{1}{(B - A)} \int_S^B (x - S) \, dx = \frac{(B - S)^2}{2(B - A)}.
\]

\[
C_O = c(S - z) + \frac{h(S - A)^2 + p(B - S)^2}{2(B - A)}.
\]

When no order is placed, the purchase cost and the reorder cost terms drop out and \( z \) replaces \( S \).

\[
C_N = \frac{h(z - A)^2 + p(B - z)^2}{2(B - A)}.
\]

Evaluating \( C_O \) with the order level equal to 112, we find that

\[
C_O = 19,729 - 110z.
\]

Expressing \( C_N \) entirely in terms of \( z \),

\[
C_N = -0.05(z - 50)^2 + 0.375(250 - z)^2.
\]

Setting \( C_O \) equal to \( C_N \), substituting \( s \) for \( z \), we solve for the optimum reorder point.

\[
19729 - 110s = -0.05(s - 50)^2 + 0.375(250 - s)^2
\]

\[
0.325s^2 - 72.5s + 3543.3 = 0
\]

Solving the quadratic\(^1\) we find the solutions

\[
s = 150.8 \text{ and } s = 72.3.
\]

The solution lying above the order level is meaningless, so we select the reorder point of 72. At this point

\[
\text{At } s = 72.3, \ C_N = C_O = 11,814.
\]

\(^1\) The solution to the quadratic \( ax^2 + bx + c = 0 \) is \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).
Since the current inventory level of 100 falls above the reorder point, no additional inventory should be purchased. If there were no fixed charge for delivery, the order would be for 12 units.

**Example 7: Demand with an Exponential Distribution**

Consider the situation of Example 6 except that demand has an exponential distribution with a mean value: $\mu = 150$. At the optimum order level

$$F(S^*) = 1 - \exp(-S/\mu) = 0.3077.$$ 

Solving for $S$,

$$S = -\mu [\ln(1 - 0.3077)] = 55.17.$$ 

The difference between $s$ and $S$ for the exponential distribution is approximately \(^2\)

$$\Delta = S - s = \sqrt{\frac{2\mu K}{c + h}} = \sqrt{\frac{2(150)(500)}{100 - 20}} = 41$$

$s = 56 - 41 = 15$.

For this distribution of demand, the current inventory of 100 is considerably above both the reorder point and the order level. Certainly an order should not be placed.

---

\(^2\) Hillier and Lieberman, page 303.