Inventory Theory.S3
Stochastic Inventory Models

There is no question that uncertainty plays a role in most inventory management situations. The retail merchant wants enough supply to satisfy customer demands, but ordering too much increases holding costs and the risk of losses through obsolescence or spoilage. An order too small increases the risk of lost sales and unsatisfied customers. The water resources manager must set the amount of water stored in a reservoir at a level that balances the risk of flooding and the risk of shortages. The production manager sets a master production schedule considering the imprecise nature of forecasts of future demands and the uncertain lead time of the production process. These situations are common, and the answers one gets from a deterministic analysis very often are not satisfactory when uncertainty is present. The decision maker faced with uncertainty does not act in the same way as the one who operates with perfect knowledge of the future.

In this section we deal with inventory models in which the stochastic nature of demand is explicitly recognized. Several models are presented that again are only abstractions of the real world, but whose answers can provide guidance and insight to the inventory manager.

Probability Distribution for Demand

The one feature of uncertainty considered in this section is the demand for products from the inventory. We assume that demand is unknown, but that the probability distribution of demand is known. Mathematical derivations will determine optimum policies in terms of the distribution.

- **Random Variable for Demand** ($x$): This is a random variable that is the demand for a given period of time. Care must be taken to recognize the period for which the random variable is defined because it differs among the models considered.

- **Discrete Demand Probability Distribution Function** ($P(x)$): When demand is assumed to be a discrete random variable, $P(x)$ gives the probability that the demand equals $x$.

- **Discrete Cumulative Distribution Function** ($F(b)$): The probability that demand is less than or equal to $b$ is $F(b)$ when demand is discrete.

\[
F(b) = \sum_{x=0}^{b} P(x)
\]

- **Continuous Demand Probability Density Function** ($f(x)$): When demand is assumed to be continuous, $f(x)$ is its density function. The probability that the demand is between $a$ and $b$ is
\[ P(a \leq X \leq b) = \int_a^b f(x) \, dx. \]

We assume that demand is nonnegative, so \( f(x) \) is zero for negative values.

- **Continuous Cumulative Distribution Function** \((F(b))\): The probability that demand is less than or equal to \( b \) when demand is continuous.

\[ F(b) = \int_0^b f(x) \, dx. \]

- **Standard Normal Distribution Function** \((\phi(x) \) and \( \Phi(x))\): These are the density function and cumulative distribution function for the Standard Normal distribution.

- **Abbreviations**: In the following we abbreviate probability distribution function or probability density function as p.d.f.. We abbreviate the cumulative distribution function as c.d.f..

**Selecting a Distribution**

An important modeling decision concerns which distribution to use for demand. A common assumption is that individual demand events occur independently. This assumption leads to the Poisson distribution when the expected demand in a time interval is small and the normal distribution when the expected demand is large. Let \( a \) be the average demand rate. Then for an interval of time \( t \) the expected demand is \( at \). The Poisson distribution is then

\[ P(x) = \frac{e^{-at} (at)^x}{x!}. \]

When \( at \) is large the Poisson distribution can be approximated with a Normal distribution with mean and standard deviation

\[ \mu = at, \text{ and } \sigma = \sqrt{at}. \]

Values of \( F(b) \) are evaluated using tables for the Standard Normal distribution. We include tables for the Standard Normal distribution at the end of this chapter.

Of course other distributions can be assumed for demand. Common assumptions are the normal distribution with other values of the mean and standard deviation, the uniform distribution, and the exponential distribution. The latter two are useful for their analytical simplicity.

**Finding the Expected Shortage and the Expected Excess**

We are often concerned about the relation of demand during some time period relative to the inventory level at the beginning of the time period. If the demand is less than the initial inventory level, there is inventory remaining at the end of
the interval. This is the condition of excess. If the demand is greater than the initial inventory level, we have the condition of shortage.

At some point, assume the inventory level is a positive value $z$. During some interval of time, the demand is a random variable $x$ with p.d.f., $f(x)$, and c.d.f., $F(x)$. The mean and standard deviation of this distribution are $\mu$ and $\sigma$ respectively. With the given distribution, we compute the probability of a shortage, $P_s$, and the probability of excess, $P_e$. For a continuous distribution

$$P_s = P\{x > z\} = \int_{z}^{\infty} f(x) \, dx = 1 - F(z), \quad (15)$$

$$P_e = P\{x < z\} = \int_{0}^{z} f(x) \, dx = F(z). \quad (16)$$

In some cases we may also be interested in the expected shortage, $E_s$. This depends on whether the demand is greater or less than $z$.

$$e_{\text{items short}} = \begin{cases} 0 & \text{if } x \leq z \\ x - z & \text{if } x > z \end{cases}.$$

Then $E_s$ is the expected shortage and is

$$E_s = \int_{z}^{\infty} (x - z) f(x) \, dx. \quad (17)$$

Similarly for excess, the expected excess is $E_e$

$$E_e = \int_{0}^{z} (z - x) f(x) \, dx.$$

The expected excess is expressed in terms of $E_s$

$$E_e = \int_{0}^{\infty} (z - x) f(x) \, dx - \int_{z}^{\infty} (z - x) f(x) \, dx$$

$$= z - \mu + E_s. \quad (18)$$

For discrete distributions, sums replace the integrals in Eq. 15 through 18.

$$P_s = P\{x \geq z\} = \sum_{x=z}^{\infty} P(x) \, dx = 1 - F(z), \quad (19)$$
\[ P_e = P \{ x \leq z \} = \sum_{0}^{z} P(x)dx = F(z). \quad (20) \]

\[ E_s = \sum_{z}^{\infty} (x-z)P(x)dx. \quad (21) \]

\[ E_e = \sum_{0}^{z} (z-x)P(x)dx = z - \mu + E_s. \quad (22) \]

In the latter two equations it doesn't matter if the \( z \) term is included, since it contributes nothing to the sums.

**When the Distribution of Demand is Normal**

When the demand during the lead time has a Normal distribution, tables are used to find these quantities. Assume the demand during the time interval has a Normal distribution with mean \( \mu \) and standard deviation \( \sigma \). We specify the inventory level in terms of the number of standard deviations away from the mean.

\[ z = \mu + k\sigma \quad \text{or} \quad k = \frac{z - \mu}{\sigma}. \]

We have included at the end of this chapter tables for the Standard normal, \( \phi(y) \), \( \Phi(y) \) and \( G(y) \). We have formerly identified the first two of these as the p.d.f. and c.d.f.. The third is defined as

\[ G(k) = \int_{k}^{\infty} (y-k)\phi(y)dy = \phi(k) - k\phi(k). \]

Using the relations between the Normal distribution and the Standard Normal it has been shown that

\[ f(z) = \frac{1}{\sigma}\phi(k) \quad (23) \]

\[ F(z) = \phi(k) \quad (24) \]

\[ E_s(z) = \sigma G(k) \quad (25) \]

\[ E_e = z - \mu + \sigma G(k) \quad (26) \]

We have occasion to use these results in the examples that follow.