Homework 5
Problems from Chapter 6 in Wolsey

Problem 1.
Show that 2-PARTITION is polynomially reducible to the 0-1 knapsack problem.

2-PARTITION: Given $n$ positive integers $(a_1, a_2, \ldots, a_n)$, find a set $S \subseteq \mathbb{N} = \{1, \ldots, n\}$ such that
$$\sum_{j \in S} a_j = \sum_{j \in \mathbb{N} \setminus S} a_j,$$
or prove that no $S$ exists.

The 0-1 knapsack problem: Given $n$ items each with a benefit $c_j$ and a weight $w_j$, find the selection of items that maximizes total benefit while staying within a weight limitation $W$. The mathematical programming model is:
$$z = \max \left\{ \sum_{j=1}^{n} c_j x_j \right\} \quad \text{subject to} \quad \sum_{j=1}^{n} w_j x_j \leq W$$
$$0 \leq x_j \leq 1 \text{ and integer}$$

Solution: Let $\sum_{j \in \mathbb{N}} a_j = b$, and assume that $b$ is even. If $b$ is odd there is no solution to the 2-partition. Let $x_j = 0$ or 1 for $j \in \mathbb{N}$ and consider the following knapsack problem.
$$z_{ip} = \max \left\{ \sum_{j \in \mathbb{N}} a_j x_j : \sum_{j \in \mathbb{N}} a_j x_j \leq \frac{b}{2}, x_j = 0 \text{ or } 1, j \in \mathbb{N} \right\}$$
Any solution yielding $z_{ip} = b/2$ is a 2-partition with $S = \{j : x_j = 1\}$. If $z_{ip} < b/2$, then no 2-partition exists.

Does this imply that the 2-partition is $NP$ Complete?
No! To show 2-Partition was a member of $NPC$, we would have to start with a known member of $NPC$, such as the 0-1 knapsack problem and reduce it to a 2-Partition.
Problem 4.
Show that SET COVERING is polynomially reducible to UFL.

Set covering problem: We are given \( n \) activities and \( m \) requirements. Each activity covers a subset of the requirements. The objective is to select a minimum cost subset of activities such that each requirement is covered at least once.

Let \( y_j = 1 \) if activity \( j \) is selected; 0 otherwise \((j \in N = \{1,\ldots,n\})\).

\[
d_j = \text{the cost of activity } j \in N
\]

\[
a_{ij} = 1 \text{ if activity } j \text{ covers requirement } i; 0 \text{ otherwise } (i \in M = \{1,\ldots,m\})
\]

Model:
\[
\text{Min. } \sum_{j \in N} d_j y_j
\]

Subject to:
\[
\sum_{j \in N} a_{ij} y_j \geq 1, \quad i \in M
\]

\[
y_j \in \{0,1\}, \quad j \in N
\]

Uncapacitated Facility Location (UFL): Given a set of potential depots \( N = \{1,\ldots,n\} \) and a set of \( M = \{1,\ldots,m\} \) clients, suppose there is a fixed cost \( f_j \) associated with the use of depot \( j \) and a transportation cost \( c_{ij} \) if any portion of the \( i \)th client’s order is delivered from depot \( j \). The objective is to decide which depots to open and from those selected, how to satisfy the total demand of each client so that the sum of the fixed costs and transportation costs are minimized.

Let \( y_j = 1 \) if depot \( j \) is opened; 0 otherwise \((j \in N = \{1,\ldots,n\})\).

\[
x_{ij} = \text{fraction of demand of client } i \text{ that is satisfied from depot } j\quad (i \in M, \quad j \in N)
\]

Model:
\[
\text{Min. } \sum_{j \in N} f_j y_j + \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij}
\]

Subject to:
\[
\sum_{j \in N} x_{ij} = 1, \quad i \in M
\]

\[
x_{ij} \leq y_j, \quad i \in M, \quad j \in N
\]

\[
0 \leq x_{ij} \leq 1, \quad y_j \in \{0,1\}, \quad i \in M, \quad j \in N
\]

Note that other formulations are possible. Also, there is an optimal solution such that each customer \( i \) is only served by a single facility. This means that \( x_{ij} = 0 \) or 1 in the optimal solution.

Reduction: Reducing the set covering problem to the UFL.
In the set covering problem, let activity \( j \) correspond to opening facility \( j \) and assume that this facility can serve all clients in the set \( S_j \subseteq M \). This means that \( a_{ij} = 1 \) if \( i \in S_j \) and 0 otherwise. Now let \( F_i \subseteq N \) be the set of facilities that can serve customer \( i \). Now set the cost \( d_j \) of activity \( j \) equal to the cost \( f_j \) of opening the facility \( j \), and set the transportation costs to 0. Finally introduce the new decision variable \( x_{ij} \) and define it as the fraction of
client $i$’s demand satisfied from facility $j$. These developments lead directly to the following instance of the UFL model above.

\[
\text{Min. } \sum_{j \in N} f_j y_j + \sum_{i \in M} \sum_{j \in N} 0x_{ij} = \sum_{j \in N} f_j y_j \\
\text{Subject to:} \\
\sum_{j \in F_i} x_{ij} = 1, i \in M \\
x_{ij} \leq y_j, i \in M, j \in N \\
0 \leq x_{ij} \leq 1, y_j \in \{0, 1\}, i \in M, j \in N
\]

The reduction effort is linear in $n + m$. The solution will have $x_{ij} \in \{0, 1\}$ for all $i$ and $j$.

Given that the set covering problem is a member of NPC, the reduction proves that UFL is also a member of NPC. For otherwise, we could solve the above instance of URL with a polynomial algorithm and then reverse the reduction in linear time to get the optimal solution to the set covering problem.
Problem 9-11 using the Teach IP Add-in.

Maximize \( z = 10x_1 + 30x_2 + 20x_3 + 20x_4 + 10x_5 \)

Subject to \( 8x_1 + 12x_2 + x_3 + 8x_4 + 2x_5 \leq 15 \)
\( 9x_1 + 7x_2 + 4x_3 + 10x_4 + 5x_5 \leq 20 \)
\( x_1 + x_2 + 8x_3 + 3x_4 + 7x_5 \leq 11 \)
\( x_j = 0 \text{ or } 1, \quad j = 1, \ldots, 5 \)

Node Level Variable Value Up/Down Visit Relax

<table>
<thead>
<tr>
<th>Node</th>
<th>Level</th>
<th>Variable</th>
<th>Value</th>
<th>Up/Down</th>
<th>Visit</th>
<th>Relax</th>
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<td>0</td>
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<td>4</td>
<td>0</td>
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<td>0</td>
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<td>1</td>
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<td>50 Integer: Replace incumbent: Backtrack Level 3 : Branch X(1) up at 1</td>
</tr>
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</table>

\( (56;0,1,1,0.2,0.2) \)

\( (53.9;0,0.5,0.9,1,0) \)

\( (45.3;0,0.7,0.9,1,0) \) Fathom

\( (54.44;0.2,1,1,0,0.3) \)

\( (48.1;0.1,1.0,4,0,1) \) Fathom

\( (52.5;0.25,1,1,0,0) \)

\( (45;0,0.5,1,0,0) \) Fathom

\( Z_B = 50 \)