1. (15 Points) Consider the linear programming model below. Variables $x_3$, $x_4$ and $x_5$ are slack variables.

Max $Z = 3x_1 - x_2$

Subject to:

\[
\begin{align*}
C1 & : 2x_1 - x_2 + x_3 = 0 \\
C2 & : -x_1 + x_2 + x_4 = 2 \\
C3 & : 2x_1 + x_2 + x_5 = 5
\end{align*}
\]

a. Sketch the feasible region on the figure below.

b. When we solve the problem we discover that for the optimal solution, $x_1$, $x_2$ and $x_4$ are basic. The dual variables for $C1$ and $C3$ are 1.25 and 0.25 respectively, while the dual variable for $C2$ is zero. Based on this information, what are the optimal values of $x_1$ and $x_2$?

$C1$ and $C3$ are the tight constraints. Solve for $x1$ and $x2$.

\[
\begin{align*}
C1 & : 2x_1 - x_2 = 0 \\
C3 & : 2x_1 + x_2 = 5
\end{align*}
\]

Subtract $C1$ from $C3$ we find $2x_2 = 5$ or $x_2 = 2.5$

From $C1$, $2x_1 = 2.5$ or $x_1 = 1.25$.

c. What does the dual variable for $C1$ (1.25) tell you? Say as much as you can without making additional calculations.

If we increase the right side of $C1$ by 1, we would expect the objective to increase by 1.25.
2. (15 Points) The table below describes a situation involving shipping between suppliers and demanders of some commodity. The rows labeled S1, S2, and S3 represent the suppliers. The numbers in the last column give the amounts available at the suppliers. The columns labeled D1, D2, and D3 represent the demanders. The numbers in the last row give the amount required by the demanders. The unit costs of shipping are shown in the body of the table. The dashes indicate disallowed shipping routes. In this problem I want algebraic LP models.

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>14</td>
<td>---</td>
<td>15</td>
<td>50</td>
</tr>
<tr>
<td>S2</td>
<td>15</td>
<td>10</td>
<td>---</td>
<td>50</td>
</tr>
<tr>
<td>S3</td>
<td>---</td>
<td>8</td>
<td>12</td>
<td>50</td>
</tr>
<tr>
<td>Demand</td>
<td>25</td>
<td>35</td>
<td>25</td>
<td>50</td>
</tr>
</tbody>
</table>

a. Write the linear programming model that will determine the minimum cost shipping plan from suppliers to demanders. The demand must be satisfied at each demander, but the entire supply need not be shipped from each supplier. Of course the amount shipped must not exceed the amount available.

Let $x_{ij} =$ amount shipped from supplier i to demander j.

Min. $14x_{11} + 15x_{13} + 15x_{21} + 10x_{22} + 10x_{32} + 12x_{33}$

Subject to:

$x_{11} + x_{13} \leq 50$
$x_{21} + x_{22} \leq 50$
$x_{32} + x_{33} \leq 50$
$x_{11} + x_{31} = 25$
$x_{22} + x_{32} = 35$
$x_{13} + x_{33} = 25$
$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$
b. Change the model in part a to allow transshipment from D1 to D2 or D3. That is, the product may be shipped first to D1 through one of the routes described by the table. Then the product may be shipped from D1 to D2 for a cost of $2 per unit or from D1 to D3 for a cost of $1 per unit.
In addition assume that 5% of the product is lost for each route used. That is if 50 units are shipped from S1 to D1 only 45 arrive at D1. Losses also occur for amounts transshipped between destinations.
Let $y_{12} =$ amount shipped from D1 to D2, and
Let $y_{13} =$ amount shipped from D1 to D3.
**Min.** $14x_{11} + 15x_{13} + 15x_{21} + 10x_{22} + 10x_{32} + 12x_{33} + 2y_{12} + y_{13}$
**Subject to:**
$x_{11} + x_{13} \leq 50$
$x_{21} + x_{22} \leq 50$
$x_{32} + x_{33} \leq 50$
$0.95x_{11} + 0.95x_{31} - y_{12} - y_{13} = 25$
$0.95x_{22} + 0.95x_{32} + 0.95y_{12} = 35$
$0.95x_{13} + 0.95x_{33} + 0.95y_{13} = 25$
$x_{ij} \geq 0$ for all $i$ and $j$, and $y_{12} \geq 0, y_{13} \geq 0$.

---

c. Say we would like to impose the requirement that only integer amounts be shipped.

In order to obtain an integer solution, must you explicitly add the integrality requirements to the model for part a? Justify your conclusion.

No. This model is equivalent to a pure network model. It is guaranteed to automatically have integer solutions.

In order to obtain an integer solution, do you have to explicitly add the integrality requirements to the model for part b? Justify your conclusion.

Yes. The model is still a network model, but the nonunity multipliers make it a generalized network. There is no guarantee of an integral solution.
3. (20 Points) A manufacturing facility as three products, A, B and C. Product demands for the next six weeks are shown in the table below.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>3</td>
<td>12</td>
<td>16</td>
<td>6</td>
<td>16</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
<td>7</td>
<td>10</td>
<td>14</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>18</td>
<td>18</td>
<td>17</td>
<td>18</td>
<td>9</td>
</tr>
</tbody>
</table>

The facility has a single production machine that operates for up to 30 hours per week. We want to schedule the machine to produce the products. The schedule is to specify how many of each product to produce in each week. Production rates for the products are shown below. Products may be produced in one week and saved in inventory for distribution in a later week. The inventory for any one of the products cannot exceed 15 units. Initial and final inventories are given in the table.

<table>
<thead>
<tr>
<th>Prod. Rate</th>
<th>Units per hour</th>
<th>Initial Inventory</th>
<th>Final Inventory</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.5</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>1.25</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Our goal is to propose a production schedule for the machine for the next 6 weeks. Our goal is to minimize total inventory. For simplicity neglect the integrality requirement for production amounts and inventory.

a. Show a network model that will solve this problem.
Inventory arcs have capacity 15. Those entering 1 and leaving 6 have upper and lower bounds equal to 10.
b. Show how the model is changed when the machine is allowed to work an additional 10 hours a week with an overtime cost of $1000 per hour above the regular cost of operation. Assume inventory is evaluated at $30 per unit per week. Now the goal is to minimize total overtime plus inventory cost. Describe how the network model will be changed to accommodate this new information.

Inventory arcs have capacity 15. Those entering 1 and leaving 6 have upper and lower bounds equal to 10. All have cost of 30.
4. (30 Points) Three products can be produced at two machining centers during a one-week period. The products may be produced in fractional amounts. The linear relationships describing this situation are listed below. The variables are:
   A, B and C are the amounts of the three products in units.
   R1 and R2 are the amounts of raw materials used in kilograms.
   T1 and T2 are the times used in the two machining centers.

   Profit: \( P = 20A + 30B + 25C - 6R1 - 8R2 \)
   Time required on machine 1: \( T1 = 5A + 8B + 10C \) (hours)
   Time required on machine 2: \( T2 = 8A + 6B + 2C \) (hours)
   Raw material 1 used: \( R1 = 1A + 2B + 0.75C \)
   Raw material 2 used: \( R2 = 0.5A + 1B + 0.5C \)
   Market Limits: \( A \leq 10, B \leq 20, C \leq 10. \)

All the variables in the above model may assume fractional values.

Say you've included all the equations and inequalities above in a linear programming model. The following paragraphs describe variations on this situation. You are complete the model by adding additional variables and constraints to the equations and inequalities described above. Show any changes or additions required. Unless stated the goal is to maximize profit. The variations are not cumulative.

   a. There is a limit of 40 hours usage per week for each machine. The total raw material use must not exceed 150 kilograms

      Add Max \( P \)
      Subject to: \( T1 \leq 40, T2 \leq 40, R1 + R2 \leq 150. \)

   b. There is a fixed cost of $100 for producing each product. That is, if we produce only one of the three products the profit is reduced by $100. If we produce two of the products, the profit is reduced by $200. If we produce all three products the profit is reduced by $300. For example if \( A = 0 \), there is no charge for that product, but if \( A > 0 \), the charge is $100.

      Add three binary variables, \( y_A, y_B, y_C \), to indicate whether a product is produced. The objective becomes:
      \( \text{Max } P - 100 \cdot y_A - 100 \cdot y_B - 100 \cdot y_C \)
      Add: \( A \leq 10 \cdot y_A, B \leq 20 \cdot y_B, C \leq 10 \cdot y_C. \)
      \( 0 \leq (y_A, y_B, y_C) \leq 1 \) and integer.
c. There is a limit of 40 hours usage per week for each machine. There is a setup time for each product that is 5 hours per setup on machine 1, and 3 hours per setup on machine 2. These setup times are the same for each machine. You must decide whether to produce or not for each product. If a product is produced, only one batch is produced in each week.

Add three binary variables, $y_A, y_B, y_C$, to indicate whether a product is produced. The machine limit constraints become:

- Time limit on machine 1: $5A + 8B + 10C + 5(y_A + y_B + y_C) \leq 40$ (hours)
- Time limit on machine 2: $8A + 6B + 2C + 3(y_A + y_B + y_C) \leq 40$ (hours)

Add: $A \leq 10 y_A$, $B \leq 20 y_B$, $C \leq 10 y_C$.

$0 \leq (y_A, y_B, y_C) \leq 1$ and integer.

d. A lucrative new market opens for product A. The new market can sell up to 10 additional units of A for revenue of $25 per unit. Management requires that the original market for A which returns a revenue of $20 per unit must be served before any units are sold in the new market.

Add a new variable, $A'$, that is the amount of the higher price units sold. Also, add a binary variable, $y$, that controls the sequence of sales. The objective becomes:

Max. Profit: $P = 20A + 25A' + 30B + 25C - 6R1 - 8R2$

Add a variable $y$. We need two new constraints:

$20y \leq A$ and $A' \leq 10y$

$0 \leq y \leq 1$ and integer, $A' \geq 0$
You are allowed to purchase more machines of each type. The number of machines must be an integer. Each machine provides a work capacity of 40 hours per week. Each machine of type 1 has a cost of $50 per week and each machine of type 2 has a cost of $75 per week.

Add integer variables:
- $y_1$: number of machine type 1 to purchase.
- $y_2$: number of machine type 1 to purchase.

Max. Profit: $P = 20A + 30B + 25C - 6R_1 - 8R_2 - 50y_1 - 75y_2$

Change time limit constraints:
- Time limit on machine 1: $5A + 8B + 10C \leq 40(1 + y_1)$ (hours)
- Time limit on machine 2: $8A + 6B + 2C \leq 40(1 + y_2)$ (hours)

Add constraints to control the order of addition.
- $y_{11} \geq y_{12}$ and $y_{21} \geq y_{22}$

$0 \leq (y_1, y_2)$ and integer
f. The raw material supplier offers the following purchase plan for raw materials. For orders below 50 units, the raw material cost is specified as given in the problem statement. For raw materials amounts above the 50, the raw materials can be purchased at a 25% discount. Note that the discount applies to all units purchased, not just the number purchased above 50.

Let plan 1 describe raw material purchases at or below 50
Let plan 2 describe raw material purchases at or above 50

Define binary variables for each raw material.
y_1 = 1 if you choose plan 1 for raw material 1, 0 otherwise.
y_2 = 1 if you choose plan 1 for raw material 2, 0 otherwise.
Note that 1 - y_1 is equal to 1 when plan 2 is selected for RM1
Note that 1 – y_2 is equal to 1 when plan 2 is selected for RM2.
Add variables R_1' and R_2' to be the amounts purchased for the lower prices.

Max. Profit:  P = 20A + 30B + 25C – 6 R_1 – 8 R_2 – 4.5 R_1' – 6 R_2'

Need to restrict purchases using the binary variables.
R_1 ≤ 50 y_1 or R_1 - 50 y_1 ≤ 0
R_1' ≤ 100(1- y_1) or R_1' + 100 y_1 ≤ 100 note that 100 can be replace by any large number
Also you must require that at least 50 be purchase if plan 2 is chosen.
50(1- y_1) ≤ R_1' or 50 ≤ R_1' + 50 y_1

Need to restrict purchases using the binary variables.
R_2 ≤ 50 y_2 or R_2 - 50 y_2 ≤ 0
R_2' ≤ 100(1- y_2) or R_2' + 100 y_2 ≤ 100 note that 100 can be replace by any large number
Also you must require that at least 50 be purchase if plan 2 is chosen.
50(1- y_2) ≤ R_2' or 50 ≤ R_2' + 50 y_2 ≥ 50

0 ≤ (y_1, y_2) ≤ 1 and integer, R_1' ≥0, R_2' ≥ 0

Raw material 1 used:  R_1+R_1' ≥ 1A + 2B + 0.75C
Raw material 2 used:  R_2 +R_2' ≥ 0.5A + 1B + 0.5C
5. (20 Points) We want to store 5 barrels of dangerous toxic materials in a storage area. The size of the area is 10 meters by 10 meters. We have identified a danger measure for each pair of the barrels. The measure is shown below with a higher number associated with a greater danger.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>9</td>
<td>10</td>
<td>18</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>0</td>
<td>14</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>14</td>
<td>0</td>
<td>23</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>6</td>
<td>23</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
<td>11</td>
<td>9</td>
<td>20</td>
<td>0</td>
</tr>
</tbody>
</table>

Each barrel is a cylinder that is 1 meter in diameter. A barrel cannot be placed closer to another or to the walls than is physically possible.

One way to express of the danger between two barrels, \( i \) and \( j \), is:

\[
c_{ij} = \frac{d_{ij}}{(x_i - x_j)^2 + (y_i - y_j)^2}
\]

Here \( d_{ij} \) is the danger measure from the table above, \( x_i \) and \( y_i \) are the coordinates of barrel \( i \) and \( x_j \) and \( y_j \) are the coordinates of barrel \( j \).

For example if barrel 1 were placed at the coordinates (0.5, 0.5) and barrel 2 were placed at (9.5, 9.5), the squared Euclidean distance between them would be \( 81 + 81 = 162 \) and the value of \( c_{12} \) would be \( 9/162 \).

Answer the following questions by providing models of the decision situation. The models are to be solved with the Excel Solver. The models cannot use functions such as ABS or Max or IF because the Solver doesn’t always work when the derivatives of a function are not continuous. As part of your solution, tell whether you think Excel will give you a global or just a local optimum.

a. Find an optimum placement of barrels to minimize the total danger. There are ten variables that determine the location of the five barrels, \( x_i \) and \( y_i \) specifying the location of barrel \( i \), for \( i = 1 \) to 5. Solve the problem:

Min. \( z = \sum_{i=1}^{4} \sum_{j=i+1}^{5} c_{ij} \)

subject to:

\( 0.5 \leq x_i \leq 9.5 \), \( 0.5 \leq y_i \leq 9.5 \) for all \( i \)

\( (x_i - x_j)^2 + (y_i - y_j)^2 \geq 1 \) for all \( i \) and \( j \)

Note that in this problem the rectangle is for 0 to 10 in both the \( x \) and \( y \) coordinates.
b. How would change the model of part a, if the floor of the storage area were in the
shape of a circle with diameter equal to 10 meters.

\[ \text{Min. } z = \sum_{i=1}^{4} \sum_{j=i+1}^{5} c_{ij} \]

subject to :
\[ (x_i)^2 + (y_i)^2 \leq (5 - 0.5)^2 = 20.25 \]
\[ (x_i - x_j)^2 + (y_i - y_j)^2 \geq 1 \text{ for all } i \text{ and } j \]

The number 20.25 is the square of the radius of the allowed circle. The
circle is 4.5 in radius because barrels can’t be put closer than 0.5 to
the wall.
Note that in this problem the variables can take on both positive and
negative values. The circle is centered at 0.

c. Forget the danger measure. What model would maximize the minimum of the Euclidean
distances between the barrels?

\[ \text{Max. } z = v \]

subject to :
\[ 0.5 \leq x_i \leq 9.5, 0.5 \leq y_i \leq 9.5 \text{ for all } i \]
\[ (x_i - x_j)^2 + (y_i - y_j)^2 \geq 1 \text{ for all } i \text{ and } j \]
\[ (x_i - x_j)^2 + (y_i - y_j)^2 \geq v \text{ for all } i \text{ and } j \]

Note that in this problem the rectangle is for 0 to 10 in both the x and
y coordinates. v is the square of the minimum distance between the
barrels.