Jensen and Bard, Chap. 2. Problems: 13 and 17. Use the Math Programming add-in and the LP/IP Solver for these problems.

13. Ten jobs are to be completed by three men during the next week. Each man works a 40-hour week. The times for the men to complete the jobs are shown in the table below. The values in the cells assume that each job is completed by a single person; however, jobs can be shared with completion times being determined proportionally. If no entry exists in a particular cell, it means that the corresponding job cannot be performed by the corresponding person.

Set up and solve a linear programming model that will determine the optimal assignment of men to jobs. The goal is to minimize the total time required to complete all the jobs.

<table>
<thead>
<tr>
<th>Man \ Task</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>—</td>
<td>7</td>
<td>3</td>
<td>—</td>
<td>—</td>
<td>18</td>
<td>13</td>
<td>6</td>
<td>—</td>
<td>9</td>
</tr>
<tr>
<td>B</td>
<td>12</td>
<td>5</td>
<td>—</td>
<td>12</td>
<td>4</td>
<td>22</td>
<td>—</td>
<td>17</td>
<td>13</td>
<td>—</td>
</tr>
<tr>
<td>C</td>
<td>18</td>
<td>—</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>—</td>
<td>19</td>
<td>—</td>
<td>8</td>
<td>15</td>
</tr>
</tbody>
</table>

Answer

**VARIABLE DEFINITIONS**

\( x_{ij} \): Proportion of job \( j \) performed by man \( i \), for \( i = A, B, C \), and \( j = 1, \ldots, 10 \).

Variables should be defined for only the man–job combinations not eliminated with an X in the table. Thus there are 20 variables in this problem.

**CONSTRAINTS**

Each man can only work 40 hours.

\[
\begin{align*}
\text{MANA:} & \quad + 7x_{A2} + 3x_{A3} + 18x_{A6} + 13x_{A7} + 6x_{A8} + 9x_{A10} \leq 40 \\
\text{MANB:} & \quad +12x_{B1} + 5x_{B2} + 12x_{B4} + 4x_{B5} + 22x_{B6} + 17x_{B8} + 13x_{B9} \leq 40 \\
\text{MANC:} & \quad +18x_{C1} + 6x_{C3} + 8x_{C4} + 10x_{C5} + 19x_{C7} + 8x_{C9} + 15x_{C10} \leq 40
\end{align*}
\]

Each job must be done.

\[
\begin{align*}
\text{TASK1:} & \quad +x_{B1} + x_{C1} = 1 \\
\text{TASK10:} & \quad +x_{A10} + x_{C10} = 1
\end{align*}
\]

**NONNEGATIVITY AND SIMPLE UPPER BOUNDS**

\( 0 \leq x_{ij} \leq 1 \) for all \( i \) and \( j \) for which variables are defined.

**OBJECTIVE FUNCTION**

MINIMIZE: Total time required for the jobs.

\[
\begin{align*}
\text{Min } Z = & \quad 7x_{A2} + 3x_{A3} + 18x_{A6} + 13x_{A7} + 6x_{A8} + 9x_{A10} \\
& \quad +12x_{B1} + 5x_{B2} + 12x_{B4} + 4x_{B5} + 22x_{B6} + 17x_{B8} + 13x_{B9} \\
& \quad +18x_{C1} + 6x_{C3} + 8x_{C4} + 10x_{C5} + 19x_{C7} + 8x_{C9} + 15x_{C10}
\end{align*}
\]
Assignment: Value 88

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Man A</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>X</td>
<td>X</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>X</td>
<td>1</td>
</tr>
<tr>
<td>Man B</td>
<td>1</td>
<td>1</td>
<td>X</td>
<td>0</td>
<td>1</td>
<td>.5</td>
<td>X</td>
<td>0</td>
<td>0</td>
<td>X</td>
</tr>
<tr>
<td>Man C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>X</td>
<td>0</td>
<td>X</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hrs.</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Man A</td>
<td>40</td>
</tr>
<tr>
<td>Man B</td>
<td>32</td>
</tr>
<tr>
<td>Man C</td>
<td>16</td>
</tr>
</tbody>
</table>
17. *(Production and Distribution)* A company operating in a third-world country has two plants, labeled A and B, which serve a major city, labeled C. The plants produce a product that will be shipped to C for sale. For discussion purposes, call a unit of the product an "item." The cost to ship one item between the plants and to the city is shown in the matrix below. The costs are given in $/unit.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>—</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>—</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>3</td>
<td>—</td>
</tr>
</tbody>
</table>

Forecasts have been made for the demand for items at the city in terms of the minimum demand (that must be satisfied), the additional demand (that may be satisfied if profitable), and the revenue per item. This information is as follows.

<table>
<thead>
<tr>
<th>Month</th>
<th>Minimum demand</th>
<th>Additional demand</th>
<th>Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>50</td>
<td>$100</td>
</tr>
<tr>
<td>2</td>
<td>120</td>
<td>70</td>
<td>$150</td>
</tr>
<tr>
<td>3</td>
<td>140</td>
<td>50</td>
<td>$130</td>
</tr>
</tbody>
</table>

The company has 60 employees at each plant. Each employee can produce 2 items per month. During the first month the employees are paid a wage of $120 per month. This amount must be paid whether or not the workers produce. Because of this low pay, the company is expecting a strike in the second month. They will not pay the employees during this month so, if the plants are to operate, other workers must be hired. The company expects that it will be necessary to pay their regular employees $150 per month after the strike. The company need not rehire all the workers if it is not profitable to do so.

To increase production during month 1 and 3, and to provide some production in month 2, the company can hire teenagers. These workers are only paid $100 per month and will produce an average of \( \frac{1}{2} \) items per month. There are as many as 100 teenage workers available. Their cost and availability remains constant over the three months.

Items can be stored at the plants as inventory. The cost of storing an item for one month is $3 at plant A and $1 at plant B. Because of environmental factors, 20% of the items stored at A at the beginning of a month are ruined during the month. The corresponding loss for plant B is 40%. For simplicity assume that all production, shipments, and sales take place at the beginning of each month.

Develop a linear programming model that when solved will determine an optimum production plan for the company over the three months. The plan is to specify for each month the number of items to produce, the number of the regular workforce to hire, the number of teenagers to hire, the quantities of shipments
between the plants, the quantities of shipments from the plants to the city, and the total amount sold in the city. Find the solution to this problem.

**Answer Distribution and Production Planning Problem**

Variable Definitions (all range from \( i = 1, 2, 3 \), except inventories that range from \( i = 0, 1, 2, 3 \)).

\[
\begin{align*}
P_{Ai} &= \text{Production at plant } A \text{ in month } i. \\
P_{Bi} &= \text{Production at plant } B \text{ in month } i. \\
S_i &= \text{Sales at } C \text{ in month } i. \\
I_{Ai} &= \text{Inventory at plant } A \text{ in month } i. \\
I_{Bi} &= \text{Inventory at plant } B \text{ in month } i. \\
x_{ABi} &= \text{shipments form } A \text{ to } B \text{ in month } i. \\
x_{ACi} &= \text{shipments form } A \text{ to } C \text{ in month } i. \\
x_{BAi} &= \text{shipments form } B \text{ to } A \text{ in month } i. \\
x_{BCi} &= \text{shipments form } B \text{ to } C \text{ in month } i. \\
W_{Ai} &= \text{Regular workers hired at plant } A \text{ in month } i. \\
W_{Bi} &= \text{Regular workers hired at plant } B \text{ in month } i. \\
T_{Ai} &= \text{Teenagers hired at plant } A \text{ in month } i. \\
T_{Bi} &= \text{Teenagers hired at plant } B \text{ in month } i. \\
V_i &= \text{Revenue per item in month } i. \\
MAX_i &= \text{Maximum sales in month } i. \\
MIN_i &= \text{Maximum sales in month } i.
\end{align*}
\]

Parameters (values are known)

\[
\begin{align*}
V_i &= \text{Revenue per item in month } i. \\
MAX_i &= \text{Maximum sales in month } i. \\
MIN_i &= \text{Maximum sales in month } i.
\end{align*}
\]

The constraints defining a general month are

**Production at the plants:**

\[
2W_{Ai} + 1.5T_{Ai} - P_{Ai} = 0.
\]

\[
2W_{Bi} + 1.5T_{Bi} - P_{Bi} = 0.
\]

**Conservation of product at the plants:**

\[
P_{Ai} + 0.8I_{Ai} - I_{Ai} - x_{ABi} - x_{ACi} + x_{BAi} = 0
\]

\[
P_{Bi} + 0.6I_{Bi} - I_{Bi} - x_{BAi} - x_{BCi} + x_{ABi} = 0
\]

**Conservation of product at the city:**

\[
x_{ACi} + x_{BCi} - S_i = 0.
\]

**Limit on Teenagers and Sales:**

\[
T_{Ai} + T_{Bi} \leq 100, \quad MIN_i \leq S_i \leq MAX_i
\]
Other limits:
\[ W_{A1} = W_{B1} = 60. \]
\[ W_{A2} = W_{B2} = 0 \]
\[ W_{A2} \leq 60, \ W_{B2} \leq 60. \]
\[ I_{A0} = 0 \]
\[ I_{B0} = 0. \]

Objective:
Maximize Profit:
\[
Z = \text{Sum}(i = 1 \ldots 3) \{ V_i S_i - 100 \tau_{Ai} - 100 T_{bi} - 3 I_{ai} - I_{bi} - 6x_{ABi} - 4x_{ACi} - 6x_{BAi} - 4x_{BCi} \} \\
- 120W_{A1} - 120W_{B1} - 150W_{A3} - 150W_{B3}
\]

<table>
<thead>
<tr>
<th>Per</th>
<th>PA</th>
<th>PB</th>
<th>S</th>
<th>IA</th>
<th>IB</th>
<th>xAB</th>
<th>xAC</th>
<th>xBA</th>
<th>xBC</th>
<th>WA</th>
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<tbody>
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</table>

Total 30971