Problem 3. (25 pts)
A crossover network is used in a loudspeaker system to channel signal components with frequencies higher than a given crossover frequency, \( f_c \), into a high-frequency speaker (tweeter) and frequencies below \( f_c \) into a low-frequency speaker (woofer). A simple representation is shown below with a signal from the amplifier being sent to a speaker.

![Diagram of a crossover network with amplifier, tweeter, woofer, and components C, L, R1, R2.]

a. An approximate equivalent circuit is shown on the right in the diagram above, where the amplifier is represented by an ideal voltage source with zero internal resistance and each speaker is represented by a resistor. If the crossover frequency is to be set as the break frequency for each channel of the speaker system (e.g., as if each is a filter), what are expressions for \( C \) and \( L \) given a value of \( f_c \). If \( R_1 = R_2 = 8\,\Omega \), and \( f_c = 1200 \,\text{Hz} \), what are the values of \( C \) and \( L \).

b. In the figure given above, clearly label the resistor that represents the tweeter and the one that represents a woofer.

c. Consider now the system as represented in the diagram below. The amplifier is now modeled with an ideal source with internal resistance, \( R_S \). Derive an expression for the total load impedance seen by the amplifier. If \( L = 2\,\text{mH} \), \( C = 125\,\mu\text{F} \), and \( R_S = 4\,\Omega \), at what frequency is maximum power transfer obtained?

![Diagram of a crossover network with amplifier, crossover circuitry, and components L, C, R1, R2.]

Note: maximum power transfer for a purely resistive circuit requires: \( R_L = R_S \). The more general case is for complex impedance relations: \( Z_L = Z_S^* \), where the * represents ‘complex conjugate’. This means the condition requires: \( R_L = R_S \) and \( X_L = -X_S \), where the \( X \) represent the complex components of the impedance.
(a) With amplifier providing an output that is an ideal voltage source, we can treat each "channel" of the speaker individually. That is, \( V \) is the input for two filters:

\[ V \xrightarrow{R_1 \ V_1} \]

High-pass filter
\[ \therefore R_1 \text{ is tweeter} \]

\[ V \xrightarrow{R_2 \ V_2} \]

Low-pass filter
\[ \therefore R_2 \text{ is woofer} \]

Ref. to Hambley, Section 6.5

\[ f_B = \frac{1}{2\pi R_c C} \]

Use the crossover frequency, \( f_c \), for \( f_B \) in both so you can find \( C \) & \( L \).

\[ C = \frac{1}{2\pi R_1 f_c} \]

\[ L = \frac{R_2}{2\pi f_c} \]

For \( R_1 = 8 \text{ \Omega} \), \( f_c = 1200 \text{ Hz} \)

\[ C = \frac{1}{2\pi (8 \text{ \Omega})(1200 \text{ Hz})} \]

\[ C = 16.58 \mu F \]

For \( R_2 = 8 \text{ \Omega} \), \( f_c = 1200 \text{ Hz} \)

\[ L = \frac{8 \text{ \Omega}}{2\pi (1200 \text{ Hz})} \]

\[ L = 1.06 \text{ mH} \]

(b) Refer to sketches above.

(c) To find total load impedance, identify this is just the equivalent impedance of speaker system (both channels).
(c) cont. Find load impedance

\[ Z_L = \frac{1}{R_1 - \frac{j}{\omega C}} + \frac{1}{R_2 + j\omega L} = \frac{(R_1 - \frac{j}{\omega C})(R_2 + j\omega L)}{R_2 + j\omega L + R_1 - \frac{j}{\omega C}} \]

\[ a_1 = (R_1 R_2 + \frac{L}{C}) - j (\frac{R_2}{\omega C} - \omega R_1 L) \]

\[ a_2 = (R_1 + R_2) + j (\omega L - \frac{1}{\omega C}) \]

Load impedance

Simplify:

\[ Z_L = \frac{[a_1 + j b_1] [a_2 - j b_2]}{[a_2 + j b_2] [a_2 - j b_2]} \]

\[ = \frac{a_1 a_2 + b_1 b_2 + j (a_2 b_1 - a_1 b_2)}{a_2^2 + j a_2 b_2 - j a_2 b_2 - (j^2) b_2^2} > a_2^2 + b_2^2 \]

\[ Z_L = \left[ \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right] + j \left[ \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right] \]

real part \quad \text{imag. part}

with \( a_1 = R_1 R_2 + \frac{L}{C} \)

\( b_1 = \frac{R_2}{\omega C} - \omega R_1 L \)

\( a_2 = R_1 + R_2 \)

\( b_2 = \omega L - \frac{1}{\omega C} \)

For max. power transfer:

\[ R_L = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} = R_S \]

\[ X_L = \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} = -X_S = 0 \]
Problem 3 (cont.)

From the \( X_L = -X_S \) relation, \( a_2b_4 = a_1b_2 \)

\[-(R_1 + R_2)(\frac{R_2^2}{\omega C} - \omega R_2) = (R_1 R_2 + \frac{1}{C})(\omega L - \frac{1}{\omega C})\]

\[-\frac{R_2^2}{\omega C} + \frac{R_2^2}{\omega C} + \omega R_1^2 L + \omega R_1 R_2 L = \omega R_1^2 L + \frac{\omega L^2}{C} - \frac{R_2^2}{\omega C} - \frac{1}{\omega C} \]

\[-R_2^2 + \omega^2 R_1^2 L C = \omega^2 L^2 - \frac{1}{\omega C} \]

\[\Rightarrow \omega = \sqrt{\frac{R_2^2 - \frac{1}{C}}{R_1^2 L C - L^2}}\]

For \( R_1 = R_2 = 8 \Omega \), \( L = 2\text{mH} \), \( C = 125\mu\text{F} \) \( \Rightarrow \omega = 2000 \text{ rad/sec.} \)

If we check \( \omega = 2000 \) in the \( R_L = R_S \) relation,

\[\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} = R_S = 4 \Omega .\]

**Note:** It turns out these equations are very sensitive to the values. If \( R_S = 5.5 \Omega \), then max power is transferred for \( 2000 \text{ rad/sec} \).

**Note:** For exam sufficient to just set up these equalities.

**Additional note:** If \( R_S \geq 4 \Omega \), max power occurs if \( C = 100 \times 125 \mu\text{F} \) and \( \omega = 200 \text{ rad/sec.} \).