Introduction

Sensors that detect acceleration are called accelerometers, and this laboratory study introduces how to set up an accelerometer for basic vibration measurement. Accelerometers are used to detect the motion of an object or point in inertial space. Note that this is different from a displacement-type sensor that senses the relative motion between two points. Interestingly, a displacement-type sensing element is required inside most accelerometers to provide a measure of displacement or force, which can then be used to infer acceleration. Accelerometers take advantage of instrumented spring-mass “seismic” structures that are designed to take advantage of Newton’s law.

Acceleration is intrinsically a dynamic variable, resulting from the application of a net force on a body. Even though the acceleration, velocity, and displacement of a body are related, it is practical to measure acceleration directly because differentiating signals from velocity or displacement sensors can introduce large errors and noise. Generally, it can be expected that low-frequency applications (on the order of 1 Hz) are best treated by displacement sensors, while intermediate frequencies (up to 1 kHz) favor velocity sensors. High-frequency measurements usually call for use of accelerometers. Nevertheless, accelerometers are finding wide application at all frequency ranges for testing and evaluating, for safety control systems, as well as within many consumer products.

All accelerometers require some means for inferring acceleration. The sensing mechanism in one of the devices used in this lab in the past was based on piezoresistive strain gauges. This accelerometer provides relatively good low frequency response and, although the calibration has a relatively low bandwidth up to 50 Hz, this is a good sensor for low frequency applications. Another accelerometer that has been used in this lab provides three axes of acceleration measurement and gives measurement down to 0 Hz (DC). This accelerometer uses a force balance approach realized using a silicon-machined capacitive-type sensing mechanism.

A good deal of effort in vibration measurement with accelerometers requires good signal conditioning and data analysis. The acceleration signal will generally be a dynamic signal proportional to the induced acceleration of the case. For example, the sensitivity for an accelerometer might be about 10 mV/g. In some applications, involving low acceleration, there may be a need to amplify the signal. In addition, there can be a slight DC offset in the output that may be due to some imbalance (e.g., if a bridge type sensor is used). This offset can change slightly over time. Some signal conditioning issues can be resolved by using an amplifier based on a differential amplifier design. This amplifier also includes an offset adjustment for removing the DC bias introduced by the imbalance in the sensor bridge. Another type of signal conditioning that can be needed is electrical filtering. Filtering can become especially important if the signal-to-noise levels get too low. Finally, many accelerometers available today require only a DC voltage excitation, and embedded electronics provide an output DC signal proportional to the acceleration in the direction of the sensitive axis.

Absolute acceleration measurements play a vital role in many feedback control systems and processes. In inertial guidance systems, accelerometers sense the general motion of spacecraft, airplanes, and missiles. In active dampers, they supply stabilizing feedback signals. In environmental and simulation tests, they help control the level and spectral content of motion. In testing and perfecting the structural behavior and performance of products, accelerometers measure the shock,
vibratory, and general motion experienced. Accelerometers also provide valuable information for monitoring the health, testing the behavior, and checking the quality of machines and structures.

This lab description provides additional information on the principle of operation of basic seismic devices, followed by two experimental studies that can be completed using equipment available in the laboratory.

**Principle of Operation: Seismic Devices / Accelerometers**

These lab exercises may make use of one of two types of accelerometers. One of the accelerometers is a piezoresistive type manufactured by Sensotec and shown in Figure 1(a). Piezoresistive accelerometers use strain gauges to detect the change in strain in elastic elements that support the seismic mass in the accelerometer. This particular device has bonded semiconductor strain gauges (wired in a full bridge) as the sensing elements.

The other accelerometer is made by Crossbow and is shown in Figure 1(b). This accelerometer relies on a capacitive sensing mechanism using silicon micromachined beams. The measurement of acceleration relies on a force balance approach, which is very different from the Flat Pack model. The accelerometer can be used to measure X, Y, and Z componentes of acceleration (a triaxial accelerometer), whereas the Flat Pack is a uniaxial accelerometer.

Both of these accelerometers can be used to measure relatively low frequency vibration, which makes them very useful for the type of motion measurements we require in this lab.

![Figure 1: Photo of laboratory configuration for studying beam-sensor.](image)

The basic element of any accelerometer is the **seismic mass**, \( m \), which is “sprung” within a case that is attached to a test object. The elastic structure (e.g., spring) that supports the mass imparts a force that will cause the seismic mass to have the same motion as the test object. A displacement-sensing element is used to infer this driving force.

By sensing force or displacement (and thus force through use of elastic element) the sensing arrangement gives an indication of the acceleration of the object to which the accelerometer casing is attached (through \( F = ma \)). Measured accelerometer signals can be integrated to provide velocity or displacement information as well, although this is not always expected to give the most favorable measurements for these quantities.
The basic structure and operation of a translational accelerometer can be modeled as a spring-mass-damper seismic structure as shown in Figure 2. The displacement of the seismic mass in response to an input case acceleration is modeled as a second-order system. Assume that the input motion is harmonic (e.g., a sinusoid), \( u(t) = u_o \sin(\omega t) \), where \( u_o \) is the amplitude and \( \omega \) is the frequency (in rad/sec). The model of the accelerometer can be found either from a direct application of Newton’s law or from the bond graph shown. Note that the variable of interest here, \( x \), is the relative displacement. The base acceleration, \( \ddot{u}(t) \) is the input in the equation,

\[
\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \ddot{u}(t),
\]

where \( x \) is the relative position, \( u(t) \), is the sinusoidal base motion, \( \zeta = b/2\sqrt{km} \) is the damping coefficient, and \( \omega_n = \sqrt{k/m} \) is the natural frequency.

A transfer function is found by letting \( s \) replace the derivative operator, \( d()/dt \), and then,

\[
\frac{X}{\hat{U}} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + \left(2\zeta \frac{s}{\omega_n}\right) + 1}.
\]

Note that here \( X \) and \( \hat{U} \) are the “s-domain” forms of \( x(t) \) and \( \ddot{u}(t) \), respectively.

If we assume that a sensing element can ideally convert the relative motion to voltage, or \( v_o = S_x \cdot x \),
then a transfer function between the output voltage signal and base acceleration is,

\[
\frac{V_o}{U} = \left[ \frac{V_o}{X} \right] \cdot \left[ \frac{X}{U} \right] = \left[ S_x \right] \cdot \left[ \frac{X}{U} \right],
\]

or,

\[
\frac{V_o}{U} = \frac{S_x}{\left( \frac{s}{\omega_n} \right)^2 + \left( 2\zeta \frac{s}{\omega_n} \right) + 1}
\]

where \( S_x \) is a sensitivity constant in volts/(units displacement).

Now, a frequency response function for the accelerometer can be found from the transfer function by substituting the frequency operator \( j\omega \) for \( s \). From a typical plot of the frequency response function (shown below) it is clear that an accelerometer accurately responds to all frequency components up to approximately \( \omega = 0.5\omega_n \). Beyond this point, there can be significant difference between the input acceleration and the output signal. The phase also increases.

![Frequency response plots for accelerometer model.](image)

Note that any dynamics introduced by the sensing element have not been included in this analysis. If there were any significant dynamics, it would be necessary to define the sensitivity function \( S_x \) more carefully. In particular, the sensitivity would be a transfer function, \( S_x = S_x(s) \), and would thus alter the relationship between output voltage and base acceleration. For this reason, it is necessary to choose a sensing element that yields a constant relationship between output voltage and displacement, for example, over the “frequency range of interest.”

The elastic sensing element, which functions to transfer deflection into an electrical signal, can be made of a variety of materials exhibiting resistive, capacitive, inductive, piezoelectric, piezoresistive, or even optical effects. Typical sensing mechanisms include:
Capacitive Accelerometers. In a capacitive accelerometer, a capacitor is formed by a “stationary” plate (the housing which moves with the base acceleration) and a moving place attached to the seismic mass. The distance between these plates determines the capacitance which can be monitored to infer acceleration.

Piezoresistive Accelerometers. Piezoresistive accelerometers use strain gauges to monitor strain in elastic elements that support the seismic mass.

Piezoelectric Accelerometers. In piezoelectric accelerometers, a crystal sensing element is used and this material can perform a dual function, acting as a precision spring as well as a sensing element. The crystal can be quartz or ceramic piezoelectric material (e.g., barium titanate, lead zirconite titanate (PZT), or lead metaniobite). The piezoelectric is sandwiched between the case and the seismic mass. The sensing material is usually designed to respond either to a compressive or shearing type force from the seismic mass, thus making the device sensitive to acceleration.

Servo Accelerometers. Servo accelerometers use an internal, closed-loop feedback control system that behaves like a mechanical spring. Measurement of acceleration occurs by virtue of a signal that generates a restoring force required to keep the seismic mass at its original position.

Experiment 1: Hanging Mass-Spring Test

Goals: to learn about accelerometers and how to use them for simple vibration measurements; to continue learning how to used computer-based data acquisition

Background reading/resources:
- Slides on Acceleration Measurement and Applications
- Discussion on accelerometers, accelerometer model and sensor mechanisms
- Note on logarithmic decrement (ME 344 note)
- Log decrement LabVIEW program (to be provided by TA)

Pre-Laboratory Preparation:

1. Review the concept of logarithmic decrement in the handout provided on the course log, and be prepared to use this concept in lab. Submit an explanation of how you use this method to determine damping ratio.
2. Review specifications (posted on the course log) for the type of accelerometer that will be used in this lab. Explain the following accelerometer specifications: range (g), sensitivity (mV/g), bandwidth (Hz).

Laboratory Work:
NOTE: This is a quick (express) type experiment, and it takes patience to set up and to conduct the experiment. The motion of the mass in directions transverse to the vertical need to minimized.

General procedure (assume guidance by TA):
1. Hang a known mass from a given spring and let it deflect under its own weight. Use a (length) scale or tape measure to make a quick check on the stiffness by noting the deflection from the spring’s free length after the mass is attached. The stiffness is roughly the weight divided by the initial deflection (for small deflection).
2. In your notebook, estimate the undamped natural frequency using the mass and stiffness. Convert from rad/sec to Hertz (cycles/sec).

3. Connect the accelerometer to the mass (there should be a velcro connection - good enough for low frequency, low g testing).

4. Make sure the excitation voltage is set to 5 volts (use a multimeter to confirm), or to the value used in the calibration and testing.

5. Run a quick unforced oscillation experiment in the following manner. Use an oscilloscope or DAQ card to measure the accelerometer signal. Deflect the mass from its equilibrium point slightly and release the mass from rest. Take care to set the mass in a motion that is primarily vertical. It is important to carefully set the initial conditions or the results may deviate significantly from what is expected. Capture 3 to 5 cycles of the acceleration signal during this unforced response experiment. Also, do not deflect the mass more than its static deflection so its weight keeps the spring in tension at all times.

6. In your notebook, record: a) the initial deflection imposed (roughly, by using a linear scale or tape measure), b) peak to peak voltage, c) period of oscillation.

7. Compare the period of oscillation measured to the undamped natural period.

8. Convert the peak to peak voltage into acceleration units (using the accelerometer sensitivity).

9. Use the initial deflection and the measurement of period of oscillation (roughly the undamped natural frequency) to compute an estimate of the peak to peak acceleration expected for this unforced response, and record all of these calculations in your notebook.

10. Comment on whether results from the experiment match the prediction. These tests are meant to confirm the acceleration measurements and to demonstrate some basic vibration calculations.

![Figure 5: Undamped oscillation experiment.](image)

Lab Evaluation (LE):

1. Determine the spring stiffness in the mass-spring setup by employing both static and dynamic measurements. Compare the two results and explain why there might be a difference.

2. Use the logarithmic decrement technique to find the effective damping ratio for the mass-spring system.

Experiment 2: Beam-Mass Vibration

Goals: to study the beam-mass vibration problem, to utilize logarithmic decrement method to find effective damping ratio, to experimentally determine the system natural frequency, and to identify
the effective mass, to reinforce understanding of 2nd order (mechanical) dynamics and vibration of a beam with mass loading the tip

Background reading:
- The concept of 'effective mass' can be understood by studying Rayleigh’s method, which shows how much mass of a spring, for example, should be included in the total mass of mass-spring system so you accurately predict natural frequency. See Appendix A: “Estimating the mass-beam natural frequency using Rayleigh’s method”.
- Be familiar with formulas in the Den Hartog table (Appendix B).

Pre-Laboratory Preparation:
1. Which formulas in the Den Hartog table(s) would you use for this lab? (Summarize your selection and reasoning).
2. Review and summarize in words how Rayleigh’s method is used in estimating the mass-beam natural frequency.
3. Outline lab procedures in your notebook for completing the proposed lab studies listed below. The TA will review your procedures and provide feedback, but you will be expected to complete the labs using your own procedure.

Proposed Lab Studies:
1. With an accelerometer mounted on the end of a strain-gauged beam (such as used in previous lab), an effective mass-spring system has been created. Experimentally determine the system damping ratio, the damped natural frequency, the undamped natural frequency, and the total (effective) mass.
2. Use your estimate of the total (effective) mass to identify the fraction of the beam’s mass that is effectively added to the tip mass. Compare this quantity with that recommended in Den Hartog’s table (see item 21 in Den Hartog formulas).

Lab Evaluation (LE):
1. Describe the procedure you developed to measure the dynamic response and to estimate the key system parameters in the system you studied.
2. Compute values for system damping ratio, the damped and undamped natural frequencies, and for the total (effective) mass and the fractional beam mass.
Estimating the mass-beam natural frequency using Rayleigh's Method.

Applying Rayleigh's method to estimate a system's natural frequency involves making some simplifying assumptions, necessary since the point is to avoid a detailed analysis of the system.

If you are interested in the natural frequency it is assumed you care about the natural vibrational mode(s) of the system. Remember for the simple mass-spring system \( mx'' + kx = 0 \), the natural frequency is simply found: \( x'' + \left( \frac{k}{m} \right)x = 0 \)

\[ \Rightarrow \text{defining natural frequency: } \omega_n = \sqrt{\frac{k}{m}} \]

This is the frequency at which this undamped system would oscillate, since the solution for \( x(0) = x_0 \) and \( x'(0) = 0 \) is \( x = x_0 \cos \omega_n t \).

Rayleigh's method for this system would proceed as follows. Find the maximum potential energy and the maximum kinetic energy for an assumed motion (or mode), then equate these and solve for the frequency of that motion. Well, we know the answer here, but just to illustrate let

\[ (P.E.)_{\text{max}} = \frac{1}{2} k \cdot x_{\text{max}}^2 \]
\[ (K.E.)_{\text{max}} = \frac{1}{2} m \cdot x_{\text{max}}^2 \]

But if the assumed motion is just \( x = x_{\text{max}} \cdot \cos \omega_n t \), then

\[ x' = x_{\text{max}} \cdot \omega_n \sin \omega_n t \]

\[ \Rightarrow x_{\text{max}} = \omega_n \cdot x_{\text{max}} \]

\[ \therefore \quad (P.E.)_{\text{max}} = (K.E.)_{\text{max}} \]
\[ \frac{1}{2} k \cdot x_{\text{max}}^2 = \frac{1}{2} m \cdot (\omega_n \cdot x_{\text{max}})^2 \]

and we see that \( \omega_n^2 = \sqrt{\frac{k}{m}} \) as expected.

The value of Rayleigh's method of course is for systems where we don't really know the "modes" exactly and we have to make assumptions which can sometimes be difficult. It is necessary to study the system and think about how it might move, statically, dynamically, and then apply the basic Rayleigh method.

For the system of a mass attached at the end of a beam, we want to know what frequency this system might naturally vibrate at when, say, given a small deflection and released.
Rayleigh's method for mass-beam (cont.) - p. 2

First, think of the two extreme cases

A Beam is ideal spring (no mass)

\[ F = k_b y \]

so

\[ \omega_n = \sqrt{\frac{k_b}{m}} \]

\[ k_b = \text{beam stiffness} = \frac{3EI}{L^3} \]

for deflection \( y \) (\( L = \text{beam length} \))

B Beam has mass, \( m_b \), which we add to end mass

\[ F = k_b m_b y \]

\[ \omega_n = \sqrt{\frac{k_b}{m + m_b}} \]

We recognize that the actual natural frequency should be between these two values:

\[ \omega_n < \omega_n \text{ actual} < \omega_n \]

In reality, we know that we should not expect that all of the beam's mass will contribute to the total effective mass of this system. A better estimate can be found by determining the fraction, \( f \), of \( m_b \) that should be added to \( m \), \( m_{\text{eff}} = m + f m_b \).

Rayleigh's method can be used to estimate this value.

Rayleigh's method for mass-beam

1. Assume natural mode follows a static cantilever deflection curve:

\[ y(x) = \frac{F x^2 (3L - x)}{6EI} \]

\[ y_{\text{max}} = \frac{FL^3}{3EI} \quad \text{at} \ x = L \]

From static beam deflection analysis.

Assume natural motion (at any \( x \)) will be

\[ y(x,t) = \tilde{y}(x) \cos \omega_n t \quad , \quad \omega_n \text{ is the unknown.} \]

2. Maximum potential energy (P.E.)max = \( \frac{1}{2} k_b y_{\text{max}}^2 \)

This is the maximum amount of potential energy stored in the system, and it occurs when the motion goes to zero (i.e., \( KE \to 0 \)).
Rayleigh's method for mass-beam - p3

2 Maximum kinetic energy - this is where you can now account for mass in the beam as well as the end mass.

\[(KE)_{\text{max}} \quad = \quad \frac{1}{2} \ m \ \bar{y}^2_{\text{max}} + (\text{max. KE due to mass in beam}).\]

First, note that this maximum value occurs when the beam is undeflected:

As it passes through the undeflected state, the beam and mass have maximum KE.

To estimate the contribution from the beam mass, use the assumed shape of \(y(x)\) from the static analysis. We need to find the accumulated KE by summing over \(x\), or

\[(KE_{\text{beam}})_{\text{max}} \quad = \quad \frac{1}{2} \ \int_{0}^{L} \left( \frac{m_b}{L} \right) \ y_n^2 \ dx\]

Assume \(y_n = y(x) \cdot w_n \cdot \sin w_n t\), so \(y_n = w_n \cdot y(x)\)

So

\[(KE_{\text{beam}})_{\text{max}} \quad = \quad \frac{1}{2} \ m_b \cdot w_n^2 \ y_{\text{max}}^2 \ \int_{0}^{L} \left[ \frac{3}{2} \left( \frac{x}{L} \right)^2 - \frac{1}{2} \left( \frac{x}{L} \right)^2 \right]^2 \ dx\]

\[= \frac{33}{140} L\]

\[\therefore (KE)_{\text{max}} \quad = \quad \frac{1}{2} \ m \ \left( w_n^2 \ y_{\text{max}}^2 \right) + \frac{1}{2} \ m_b \cdot \frac{33}{140} \ \left( w_n^2 \ y_{\text{max}}^2 \right) \quad \text{(total)}.\]

4 Now, equate: \((PE)_{\text{max}} = (KE)_{\text{max}} \text{ (total)}\) (Rayleigh's method)

\[\frac{1}{2} \ k_b \ y_{\text{max}}^2 \quad = \quad \frac{1}{2} \left[ m + \frac{33}{140} \ m_b \right] \cdot w_n^2 \ y_{\text{max}}^2\]

\[\therefore \quad w_n \quad = \quad \sqrt{\frac{k_b}{m + \frac{33}{140} \ m_b}} \quad \frac{33}{140} \quad \approx \quad 0.236\]

Compare to item (21) in Den Hartog table of formulas.
A COLLECTION OF FORMULAS

I. Linear Spring Constants

(“Load” per inch deflection)

Coil dia. $D$; wire dia. $d$; $n$ turns

\[ k = \frac{Gd^4}{8nD^3} \]  

(1)

Cantilever

\[ k = \frac{3EI}{l^3} \]  

(2)

Cantilever

\[ k = \frac{2EI}{l^2} \]  

(3)

Beam on two supports; centrally loaded

\[ k = \frac{48EI}{l^3} \]  

(4)

Beam on two supports; load off center

\[ k = \frac{3EI}{l_1^2l_2^2} \]  

(5)

Clamped-clamped beam; centrally loaded

\[ k = \frac{192EI}{l^3} \]  

(6)

Circular plate, thickness $t$; centrally loaded; circumferential edge simply supported

\[ k = \frac{16\pi D}{R^2} \frac{1 + \mu}{3 + \mu} \]  

in which the plate constant is

\[ D = \frac{Et^3}{12(1 - \mu^2)} \]  

(7a)

\[ \mu = \text{Poisson’s ratio} \approx 0.3 \]

Circular plate; circumferential edge clamped

\[ k = \frac{16\pi D}{R^2} \]  

(8)

Two springs in series

\[ k = \frac{1}{1/k_1 + 1/k_2} \]  

(9)

II. Rotational Spring Constants

(“Load” per radian rotation)

Twist of coil spring; wire dia. $d$; coil dia. $D$; $n$ turns

\[ k = \frac{Ed^4}{64nD} \]  

(10)
Bending of coil spring

\[ k = \frac{Ed^4}{32nD} \cdot \frac{1}{1 + E/2G} \]  

(11)

Spiral spring; total length \( l \); moment of inertia of cross section \( I \)

\[ k = \frac{EI}{l} \]  

(12)

Twist of hollow circular shaft, outer dia. \( D \), inner dia. \( d \), length \( l \)

\[ k = \frac{Gl_p}{l} = \frac{\pi}{32} \frac{G(D^4 - d^4)}{l} \]  

(13)

For steel \( k = 1.18 \times 10^6 \times \frac{D^4 - d^4}{l} \)

Cantilever

\[ k = \frac{EI}{l} \]  

(14)

Cantilever

\[ k = \frac{2EI}{l^2} \]  

(15)

Beam on two simple supports; couple at center

\[ k = \frac{12EI}{l} \]  

(16)

Clamped-clamped beam; couple at center

\[ k = \frac{16EI}{l} \]  

(17)

III. Natural Frequencies of Simple Systems

End mass \( M \); spring mass \( m \), spring stiffness \( k \)

\[ \omega_n = \sqrt{k/(M + m/3)} \]  

(18)

End inertia \( I \); shaft inertia \( I_s \), shaft stiffness \( k \)

\[ \omega_n = \sqrt{k/(I + I_s/3)} \]  

(19)

Two disks on a shaft

\[ \omega_n = \sqrt{k(I_1 + I_2)} \]  

(20)

Cantilever; end mass \( M \); beam mass \( m \), stiffness by formula (2)

\[ \omega_n = \sqrt{\frac{k}{M + 0.23m}} \]  

(21)

Simply supported beam; central mass \( M \); beam mass \( m \); stiffness by formula (4)

\[ \omega_n = \sqrt{\frac{k}{M + 0.5m}} \]  

(22)

Massless gears, speed of \( I_2 \) \( n \) times as large as speed of \( I_1 \)

\[ \omega_n = \sqrt{\frac{1}{k_1} + \frac{1}{n^2k_2}} \times \frac{I_1 + n^2I_2}{I_1 \cdot n^2I_2} \]  

(23)

\[ \omega_n = \pm \left( \frac{k_1 + k_2 + k_1 + k_3}{I_1 + I_3} \right) \pm \frac{I_1I_3}{I_1I_2I_3} (I_1 + I_2 + I_3) \]  

(24)