Linear Programming
Example Qualifying Exam Questions

For each of the problems, full points will only be given for correct, complete, understandable, and unambiguous answers.

Note: These questions total 50% in value because the other half of the optimization qualifier concerns integer programming.

1. (25%) The simplex method for an LP with bounded variables solves a problem of the form:
\[
\begin{align*}
\min_x & \quad cx \\
\text{s.t.} & \quad Ax = b \\
& \quad \ell \leq x \leq u,
\end{align*}
\]
where \(\ell \leq u\). We place no sign restrictions on \(\ell, u,\) or \(b\). A bounded basic feasible solution may be represented by the partition \(A = (B, N_\ell, N_u)\) and \(x = (x_B, x_{N_\ell}, x_{N_u})\) where \(x_{N_\ell} = \ell, x_{N_u} = u,\) and \(x_B = B^{-1}(b - N_\ell \ell - N_u u)\) satisfies \(\ell_B \leq x_B \leq u_B\).

Consider the following phase-I LP for the bounded variable formulation:
\[
\begin{align*}
\min_{x,y} & \quad ey \\
\text{s.t.} & \quad Ax + Iy = b \\
& \quad \ell \leq x \leq u \\
& \quad 0 \leq y,
\end{align*}
\]
where \(e = (1, 1, \ldots, 1)\).

(a) Let \(X = \{x : Ax = b, \ell \leq x \leq u\}\). Let \((x^*, y^*)\) be an optimal solution to the phase-I LP. Show that \(X \neq \emptyset\) if and only if \(y^* = 0\).

(b) Construct an initial bounded basic feasible solution to the phase-I LP under the assumption that \(b - A\ell \geq 0\).

(c) Explain how to create an initial bounded basic feasible solution for the phase-I LP when some rows violate \(b_i - A_i \ell \geq 0\). (You may modify the original problem as long as your transformation yields an equivalent LP.)

2. (25%) Consider the following linear program:
\[
\begin{align*}
\min_x & \quad cx \\
\text{s.t.} & \quad Ax = b : \pi \\
& \quad x \geq 0.
\end{align*}
\]
Assume that the above primal LP and its dual LP each have unique nondegenerate optimal solutions \(x^*\) and \(\pi^*\), respectively.

(a) Suppose it is known in advance that the \(j^{th}\) component of \(x^*, x^*_j\), has value 0, i.e., \(x^*_j\) is a nonbasic variable in the optimal solution. Describe how this information can be used to solve a smaller linear program.

(b) Describe how the solution to the reduced LP from (a) can be used to verify that the assumption, \(x^*_j = 0\), is valid.

(c) Suppose, instead, it is known in advance that \(x^*_j\) has a positive value but that the specific value for \(x^*_j\) is not known. Describe how this information can be used to solve a smaller linear program.

(d) Describe how the solution to the reduced LP from (c) can be used to find \(x^*_j\) and verify that the assumption, \(x^*_j > 0\), is valid.
### Integer Programming

**Sample PhD Qualifier Questions**

1. Let \( X = \{(x, y) \in \mathbb{R}^2 \times Z^2 : 3x_1 + x_2 + 2y_1 + 4y_2 \geq 13\} \). Given the feasible point \((x', y') = (0, 0.6, 3.1, 1.55)\), find a valid inequality for \( X \) that cuts off this point.

2. (Branch and bound can take exponential time) Consider the 0-1 programming problem

   \[
   \text{Minimize} \quad x_{n+1} \\
   \text{subject to} \quad 2x_1 + 2x_2 + \cdots + 2x_n + x_{n+1} = n \\
   \quad x_j \in \{0, 1\}, \quad j = 1, \ldots, n
   \]

   Show that any branch and bound algorithm that uses linear programming relaxations to compute lower bounds, and branches by setting a fractional variable to either 0 or 1, will require the enumeration of an exponential number of nodes when \( n \) is odd.

   **Hint:** It might be useful to set \( n = 2k+1 \) on the right-hand side of the constraint. Think about what \( k \) signifies in terms of the LP relaxation.

3. **Matching.** Given an undirected graph \( G = (V, E) \) with node set \( V \) and edge set \( E \), a 1-matching denoted by \( M \) is a subset of \( E \) with the property that each node in the subgraph \( G(M) = (V, M) \) is met by no more than one edge. To develop an optimization model, let \( x_e = 1 \) if edge \( e \) is included in the matching and 0 otherwise, and let \( c_e \) be the weight of edge \( e \).

   Also, let \( \delta(v) \) be the set of edges incident to node \( v \in V \) and let \( E(S) \) be the set of edges between the nodes in \( S \subseteq V \).

   a. Using this notation, write an integer programming model for the maximum weighted matching problem on \( G \). **Hint:** only one set of constraints is needed.

   b. Let \( S \subseteq V \) be an odd set; that is, \(|S| \geq 3\) and odd. Write another set of constraints for the problem that generalizes the observation that if \(|S| = 3\), then only one edge in \( E(S) \) can be in the matching. These constraints are similar to the subtour elimination constraints in the TSP.
c. If you are told that the constraints from parts (a) and (b) define the convex hull of 1-matchings, does this indicate that the maximum weighted matching problem is in \( \mathcal{P} \)?

Explain.

3. \( \text{QAP} \). Let \( d_{ijpq} \) be a nonnegative constant for \( i,j,p,q = 1,…,n \), and let \( x_{ip} \in \{0,1\} \) for all \( i \) and \( p \). The quadratic assignment problem is

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} d_{ijpq} x_{ip} x_{jq} \\
\text{subject to} & \quad \sum_{p=1}^{n} x_{ip} = 1, \quad i = 1,…,n \\
& \quad \sum_{i=1}^{n} x_{ip} = 1, \quad p = 1,…,n \\
& \quad x_{ip} \in \{0,1\}, \quad i = 1,…,n, \quad p = 1,…,n
\end{align*}
\]

a. Is QAP in \( \mathcal{NP} \)? Explain.

b. Show that the TSP can be polynomially reduced to an instance of the QAP. Define all additional notation used in your solution.

**Hint:** consider a TSP formulation in which the decision variable represents the position of a node in a tour.

c. What does this tell you, if anything, about the computational complexity of QAP?

Explain.