Key Problem 1

Assume steady, incompressible, Apply
Bernoulli from 1-2 and from 1-3:

1-2:

\[ p_1 + \frac{\rho V_1^2}{2} + \rho g z_1 = p_2 + \frac{\rho V_2^2}{2} + \rho g z_2 \]

\[ V_1 = \frac{Q_1}{A_1} = 10 \text{ m/s} \]

\[ p_2 = p_1 + \frac{\rho}{2} (V_1^2 - V_2^2) = 300 \frac{\text{kN}}{\text{m}^2} + \frac{99.8}{2} \frac{\text{m}}{\text{s}^2} \left(10 - 14^2\right) \]

\[ p_2 = 252 \text{ kPa} \]

2-3:

Need to know \( V_2 \). Apply continuity:

\[ Q_1 = Q_2 + Q_3 \rightarrow Q_2 = \frac{1}{m^3/s} - (14 \times 0.03) = 0.58 \frac{m^3}{s} \]

\[ V_2 = \frac{Q_2}{A_2} = 16.57 \text{ m/s} \]

Bernoulli:

\[ p_1 + \frac{\rho V_1^2}{2} + \rho g z_1 = p_3 + \frac{\rho V_3^2}{2} + \rho g z_3 \]

\[ p_3 = p_1 + \frac{\rho}{2} (V_1^2 - V_2^2) - \rho g z_3 \]

\[ = 300 \times 10^3 \frac{\text{N}}{\text{m}^2} + \frac{99.8}{2} \frac{\text{m}}{\text{s}^2} \left(10 - 16.57^2\right) - (998)(9.8)(10) \]

\[ = 300 \times 10^3 - 87.1 \times 10^3 - 97.8 \times 10^3 \]

\[ p_3 = 115 \text{ kPa} \]
Key - Prob 2

\[ \nabla = (A \lambda + B) \hat{\lambda} - A \hat{\lambda} \]

\[ \frac{\partial A}{\partial x} = B = 0 \]

(a) Steady \ (\text{neither} \ u \text{ nor} \ v = f(t))

(b) Does it satisfy continuity?

For steady, comp flow: \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

\[ \frac{\partial u}{\partial x} = A \quad \frac{\partial u}{\partial y} = -A \quad \frac{\partial w}{\partial z} = 0 \quad (w = 0 \text{ everywhere}) \]

So continuity is satisfied. \ OK.

(c) For irrotationality, \( \omega = 0 \) everywhere.

\[ \omega = \frac{1}{2} \left[ \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \right] + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]

\[ \frac{\partial w}{\partial y} = 0 \quad \frac{\partial v}{\partial z} = 0 \Rightarrow \omega_x = 0 \]

\[ \frac{\partial u}{\partial z} = 0 \quad \frac{\partial w}{\partial x} = 0 \Rightarrow \omega_y = 0 \]

\[ \frac{\partial v}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 0 \Rightarrow \omega_z = 0 \]

Irrotational.

(d) \[ \alpha = \frac{\partial \tau}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial \tau}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial \tau}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial \tau}{\partial z} \]

\[ \alpha_x = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = (A \lambda + B)(A) + (-A)(0) = A^2 + AB \]

\[ \alpha_y = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = (A \lambda + B)(0) + A \lambda (-A) = A^2 \]

Plugging in \( u(x, y) = (3, 2) \)

\[ \alpha_x = 100(3) + 30 = 330 \text{ ft/s}^2 \]

\[ \alpha_y = 100(2) = 200 \text{ ft/s}^2 \]
\[ H = f(L, \epsilon, \varphi, a, d, D, q) \quad N = 7 \]
\[ \epsilon = \frac{M}{L^2} \quad \varphi = \frac{L}{T} \]
\[ M \times 3 \quad 4 \pi \text{ groups} \]

(a) \( \epsilon = [M^1 L^{-2}] \quad d = [L] \quad \varphi = [T^{-1}] \)

OK because all units \((M, L, T)\) are represented.

(b) Need 4 \( \pi \) groups with \( \epsilon, d, \varphi \) as repeating; \( H, \epsilon, a, D, q \) non-repeating.

By misreporting:

\[ \Pi_a = \frac{\epsilon}{ \varphi} \quad \Pi_H = \frac{H}{d} \quad \Pi_D = -d \]

Determine \( \Pi_g \):

\[ \Pi_g = [M^0 L^0 T^0] = \epsilon_a d^b \varphi^c g^1 = \left[ \frac{M^1}{L^2} \right] \left[ \frac{L}{T} \right] \left[ \frac{T}{L^2} \right] \]

\( M : \ 0 = a \)

\( L : -3a + b + c + 1 = 0 \Rightarrow b = +1 \)

\( T : 0 = -c - 2 \Rightarrow c = -2 \)

\[ \Pi_g = \frac{3d}{\sqrt{2}} \]

(c) \( \Pi_g_m = \Pi_g_p \Rightarrow \frac{\delta m}{V_m} = \frac{\delta p}{V_p} \)

\[ \frac{V_m^2}{V_p^2} = \frac{dm}{dp} - \frac{1}{100} \Rightarrow V_m = V_p \sqrt{\frac{1}{100}} \]

\[ V_m = \frac{1}{10} V_p = 3 \text{ m/s} \]

\[ \text{and} \quad H_p = 100 \quad H_m = 3 \text{ m} \]
4. (20 points) Water enters a plane channel which is infinitely wide in the x direction with uniform velocity and flows steadily downward under the action of gravity. Starting with the full Navier-Stokes equations for incompressible flow in Cartesian coordinates shown below, list the assumptions you can make about this flow and show what terms can be eliminated to reduce the equations to their simplest form for determining the pressure and velocity distribution in the channel. List the assumptions by number and put the number next to the slash mark for each term you eliminate.

\[
\begin{align*}
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) &= \rho g_x \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) &= -\rho g_y + \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) &= \rho g_z + \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\end{align*}
\]

Assumptions:
1. Steady flow \( \Rightarrow all \ \frac{\partial}{\partial t} = 0 \)
2. Two-dimensions \( \Rightarrow all \ \frac{\partial}{\partial x} = 0 \)
3. No \( g_x \) or \( g_z \). \( g_y = -g \)
4. Since \( U = 0 \) at entrance and \( \frac{\partial u}{\partial x} = 0 \) everywhere, \( U = 0 \) everywhere (symmetry)