OPTIMAL HEDGE FUND ALLOCATION WITH
ASYMMETRIC PREFERENCES AND DISTRIBUTIONS

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1 Abstract

Hedge funds typically have non-normal return distributions marked by significant positive or negative skewness and high kurtosis. Mean variance models ignore this, and thus fail to give an accurate view of how hedge funds will work in a portfolio. We propose a method that uses new optimization methodology to solve for a variety of investment objectives and address the special issues of hedge fund allocation. The method incorporates a combination of Monte Carlo simulation and optimization that solves the different objectives. We applied the model to examine the effects of semi-variance, conditional third and fourth moments on portfolio allocation with hedge funds. The results indicate a substantial allocation to hedge funds is justified even with consideration for the highly unusual kurtosis and skewness.

2 Introduction

In the last decade, the investment styles collectively known as hedge funds or absolute return strategies have grown in number, assets and specialization. High net worth investors as well as endowments and foundations discovered early the value of including hedge funds in their plans. Corporate and public plans have only recently become interested. They all face the same challenge of how to effectively incorporate hedge funds into their optimum mix.

Hedge funds typically have non-normal return distributions marked by significant positive or negative skewness and high kurtosis. Mean variance models ignore this, and thus fail to give an accurate view of how hedge funds will work in a plan. In simple terms, mean variance models penalize funds that occasionally surprise on the upside while they also underestimate the risk of the funds that have asymmetric downside risk.

The traditional asset classes have been studied extensively and long detailed data histories are available to support their study. The data indicates that the return series are close to normal, that is; they can be defined well by simply their mean return and standard deviation.

Insert Figure 1 Here
Hedge fund performance is not so simply described. The monthly return series for hedge funds demonstrate skewness and kurtosis. Two examples of large hedge funds (assets over $1 billion with history of greater than five years) are shown below.

Index data for hedge funds present complications beyond their non-normal distributions. The source of the other problems is the lack of consistency and rigor in most indices.

Returns are likely to be higher than actuals as a result of "instant histories", survivor bias and self-selection. In several of the indices, the constituents are determined by what funds decide to list in a database. When a fund does list, it typically brings 18 to 24 months of history into the index, and probably a good history. On the other hand, some of the biggest funds do not choose to list and their information is not included in the index. At the other end of the equation, when funds stop reporting data, the months thereafter are not factored into the index. Survivor bias is estimated to add 1.5% to 3.0% to annual returns. Further the index itself may not be truly investable, so the potential is inaccurately represented in an asset allocation project.

While various factors increase observed returns, volatilities are likely reported lower than actuals as a result of autocorrelation. The combination of higher observed returns and lower observed standard deviations result in higher Sharpe ratios, the primary driver of results in mean-variance optimizations. Adding in the skewness and kurtosis characteristics, mean-variance models can be expected to over-allocate to hedge funds.

The Markowitz mean-variance approach was introduced in the early 1950s as a rational tool to help guide the decision on optimal portfolio allocation. One of its basic assumptions is that the investor’s objective is defined as a trade-off between risk and return.

This methodology was used to determine what is often considered the baseline neutral asset allocation: 60% stocks and 40% bonds. The portfolio represents the maximum Sharpe ratio portfolio in a mean-variance optimization where the input statistics come from the long run.
averages computed from the monthly returns for US equity (STOCKS), US long-term bonds (BONDS) and US 30 days T-bills (CASH) for the period 1926 to 2000:

<table>
<thead>
<tr>
<th></th>
<th>STOCKS</th>
<th>BONDS</th>
<th>CASH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>13%</td>
<td>6%</td>
<td>3.85%</td>
</tr>
<tr>
<td>Volatility</td>
<td>20.20%</td>
<td>8.70%</td>
<td>0.92%</td>
</tr>
</tbody>
</table>

With this example, we can see that if the investor has the traditional objective of seeking to maximize returns per unit of risk, the investor can use the mean-variance approach for portfolio allocation. The framework is less useful, however, if the investor’s preference is different or if the return series is not normally distributed, as is the case with hedge funds.

Some of the restrictive assumptions of the mean-variance approach are no longer necessary. During the last 10 years, researchers have made substantial advances in the field of mathematical optimization in seeking solutions to complex planning issues in such fields as transportation, communications and manufacturing. These advances in optimization methodologies combined with Monte Carlo simulation techniques and the rapid advances in computing power in recent years have created powerful new means to help make optimal decisions under uncertainty (an excellent source in this area is Pflug [8]).

We propose a method that uses the new effective optimization methodology to solve for a variety of investment objectives and address the special issues of hedge fund allocation.

3 Optimization Model for Utility Maximization

We used these breakthroughs to develop a robust and flexible framework for asset allocation. The optimization techniques allow you to solve for allocations to meet a broad range of investment objectives. The simulation techniques discussed in the next section allow the model to capture the information contained in the skewness and kurtosis of the non-normal return distributions.

The most general preference function is often expressed as maximizing expected utility, that is, maximizing the value to the investor. While the traditional objective of maximizing risk-adjusted returns is useful in many contexts, investors often have other preferences. An
institution may want to increase the probability that it can achieve a certain benchmark return in order to meet the liabilities of its pension plan. Or, it may want to minimize the need to make additional contributions to a plan.

Problems that find optimal allocation with respect to a set of constraints by taking into account the uncertainty of the underlying asset returns are known as stochastic programs. Next, we will formally introduce such a problem.

Maximizing expected utility is typically expressed as: \( U(X, \tilde{\omega}) \). An investor in a world of uncertain returns can choose from a universe of assets \( X = \{X_1, X_2, \ldots, X_n\} \). The assets are assumed to have random returns. The problem is to determine the proportion to be invested into each asset. Portfolio weights are denoted by \( \tilde{\omega} = \{\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n\} \), \( \tilde{\omega} \in W \). They might be subjected to linear, quadratic, or other types of constraints, such as cardinality. These might arise when the investor would like to choose a subset of the original universe but still maintain the initial constraints.

Putting the components together, the generic problem (optimizing over one period) can be expressed as follows:

\[
\begin{align*}
    z^* &= \max_{\tilde{\omega} \in W} \mathbb{E} \left[ U(X, \tilde{\omega}) \right] \\
    \text{s.t.} & \quad \text{Risk Constraint} \\
                    & \quad \text{Additional Constraints}
\end{align*}
\]

This generic formula can be further refined to express more specific objectives or preferences. For example, if you are interested in maximizing the probability of outperforming a benchmark, \( r \), the problem can be expressed in terms of maximizing expected utility as follows:

\[
    z^* = \max_{\tilde{\omega} \in W} \mathbb{E}_X \left[ I \left( \tilde{\omega}^T X > r \right) \right]
\]

In this formula, \( I \) denotes the indicator function which takes a value of 1 if the event in the brackets occurs or 0 otherwise.

The formula can also be expressed easily in terms of maximizing probability:

\[
    z^* = \max_{\tilde{\omega} \in W} P \left( \tilde{\omega}^T X > r \right) \quad \quad (1)
\]
A different objective might be to minimize expected shortfall, that is to minimize the average size of any shortfall in those instances when the portfolio fails to outperform the benchmark \( r \). In utility terms, this is expressed as follows:

\[
    z^* = \min_{\bar{\omega} \in W} E_X \left[ r - \bar{\omega}^T X \right]^+ \tag{2}
\]

Compound objectives can also be established for the model. Combining (1) and (2) above, the investor’s objective might be to maximize the probability of outperforming a benchmark, and at the same time wanting to limit the average amount by which he would underperform benchmark. This combination objective can be formulated as:

\[
    z^* = \max_{\bar{\omega} \in W} P \left( \bar{\omega}^T X > r \right) - \lambda E_X \left[ r - \bar{\omega}^T X \right]^+ \tag{3}
\]

Formula (3) represents the objective of an investor, who has two benchmarks,

1. \( r_1 \) is the benchmark he wants to outperform, for example, the return of the S&P500
2. \( r_2 \) is the benchmark he does not want to fall below, for example zero or the risk free rate.

The value of \( \lambda \) represents the priority or the weight placed on the second objective. By varying that weight, we can obtain a frontier in much the same way as one constructs the mean-variance efficient frontier by varying the desired expected return or the risk aversion parameter of the investor. However, in this frontier the axes are probability of outperformance and size of the average conditional shortfall.

### 4 Solution Methodology

Problems (1), (2) and (3) are one-period stochastic programming problems. They can be solved exactly if \( X \) has a small number of scenarios, or if \( X \) is from a normal distribution in which case, the problems can be simplified to deterministic nonlinear programming problems.
If $X$ is from a non-normal distribution and the number of scenarios is either large or continuous, then it is usually not possible to simplify the stochastic problem and solve it exactly. The only available approach is to solve the problem approximately.

Asset allocation problems with hedge funds present, generally involve both non-normal distributions and a continuous range of potential returns. Thus, the only available approach is to solve the problem approximately. The developments in the field of mathematical optimization combined with the rapid advances in computing power make possible the ability to generate reliable approximate solutions.

One standard approach for approximately solving stochastic programs is to use Monte Carlo sampling procedures. This method uses computer programs to generate randomly a number of observations, $N$, of some variable, $X$, from some distribution of potential values, and then solve the approximate stochastic program using that information. This solution with the sampled data represents one possible realization of the problem. By doing this many times, a series of approximate solutions can be constructed. These solutions can be used to construct an upper bound of the objective function. It will be an “upper bound” since we have perfect information about future returns and create the optimal portfolio weightings with that perfect information. That is, we "see the future" and "cheat" when making a decision regarding our portfolio allocation. This upper bound is known as a "wait and see" bound.

We can also compute a "lower bound" of the objective function. For any portfolio weights, $\bar{\omega}$, generate a large number of observations of the monthly returns for each asset class, and estimate the value for the objective function. This bound is known as "here and now" bound, since we start with a feasible weighting and then simulate what may take place with that weighting.

By employing these two bounding ideas together, a confidence interval on the optimality gap (the difference between the optimal value of the objective function, $z^*$, and the value for a given solution vector, $\bar{\omega}$) can be constructed. This confidence interval represents a statement of the quality of the solution $\bar{\omega}$. This methodology was originally developed in Mak, Morton and Wood [5], where they applied it to a variety of operations research problems. For a detailed
description of the approximating model and new approaches for solving it see Popova, Morton and Popova [9].

5 Convergence of the Approximate Solution

Stochastic programs address decision-making with uncertainty around the variables. As we discussed, some of them can be solved only approximately. To test such methodologies, we run a large number of scenarios for a case with specified distributions, for example, a normal distribution. The results of the sequence of tests are then compared with an exact solution found using another proven methodologies. Mak, Morton and Wood [5] demonstrated that, as the number of scenarios increases the lower and upper bounds on the objective function should converge toward the true optimal value.

Here we look at a universe of 6 asset classes. We assume that the underlying true distribution is normal $N(\mathbf{EX}, \Sigma)$ and we know the expected return, $\mathbf{EX}$ (the vector with the expected values of the asset classes) and the covariance matrix, $\Sigma$. For this particular example we use the historical returns and covariance matrix for 6 asset classes:

$$\mathbf{EX} = (15.8\%, 8.3\%, 6.5\%, 44.3\%, 9.1\%, 15.2\%)$$

$$\Sigma = \begin{bmatrix}
1.91\% & 0.21\% & 0.15\% & 1.68\% & 1.33\% & 0.26\% \\
0.21\% & 0.16\% & 0.12\% & 0.03\% & 0.09\% & 0.04\% \\
0.15\% & 0.12\% & 0.40\% & -0.06\% & 0.01\% & -0.01\% \\
1.68\% & 0.03\% & -0.06\% & 6.40\% & 2.04\% & 0.55\% \\
1.33\% & 0.09\% & 0.01\% & 2.04\% & 2.29\% & 0.28\% \\
0.26\% & 0.04\% & -0.01\% & 0.55\% & 0.28\% & 0.21
\end{bmatrix}$$

When the returns come from a multivariate normal distribution with known parameters and the objective function is to maximize probability of outperforming a fixed benchmark, the optimal portfolio is the one that maximizes the information ratio. For our example, we make typical assumptions: the fixed benchmark is 10%, short selling is not allowed and the sum of the weights must equal one. The resulting quadratic programming problem can be solved with commercially available software, for example CPLEX [4]. With the above data, the optimal portfolio is:
Since we are assuming normality, the probability of outperformance also can be computed "exactly", see Popova, Morton and Popova [9] as follows:

\[
P(\bar{\omega}^T X > 10\%) = 1 - \Phi \left[ \frac{(10\% - \bar{\omega}^T E X)}{\sqrt{\bar{\omega}^T \Sigma \bar{\omega}}} \right]
\]

\[
= 92.88\%
\]

where, \( \Phi \) denotes the standard normal cumulative function.

We then use our methodology to solve two sequences of approximate stochastic programs using the same data to establish the upper and lower bounds. We test with 100 scenarios, 250 scenarios, 500 scenarios and then 1,000 scenarios as shown in the table below.

<table>
<thead>
<tr>
<th>MAX INFORMATION RATIO</th>
<th>True Probability</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>99% CI on the Optimality Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 SCENARIOS</td>
<td>92.88%</td>
<td>92.20%</td>
<td>94.87%</td>
<td>[0.0%; 2.9%]</td>
</tr>
<tr>
<td>250 SCENARIOS</td>
<td>92.52%</td>
<td>92.55%</td>
<td>94.00%</td>
<td>[0.0%; 1.7%]</td>
</tr>
<tr>
<td>500 SCENARIOS</td>
<td>92.55%</td>
<td>92.62%</td>
<td>93.16%</td>
<td>[0.0%; 0.8%]</td>
</tr>
<tr>
<td>1000 SCENARIOS</td>
<td>92.62%</td>
<td>92.62%</td>
<td>93.16%</td>
<td>[0.0%; 0.8%]</td>
</tr>
</tbody>
</table>

The lower bound and the upper bound both converge on the "true" probability as the number of scenario increases. In addition, the last column shows that the size of the 99% confidence interval on the optimality gap decreases from almost 3% to less than 1% as the number of scenario increases.

Since all the weights are bounded between 0 and 1, one can expect to see convergence not only in the value function, that is, the probability of outperformance, but also in the solution
space. The next table illustrates this convergence behavior. As the number of scenario increases, the optimal allocation converges toward the "true" solution.

<table>
<thead>
<tr>
<th>ASSET CLASS</th>
<th>Analytical solution</th>
<th>100</th>
<th>250</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLASS 1</td>
<td>0.00%</td>
<td>2.41%</td>
<td>0.98%</td>
<td>0.81%</td>
<td>0.55%</td>
</tr>
<tr>
<td>CLASS 2</td>
<td>0.00%</td>
<td>2.07%</td>
<td>2.17%</td>
<td>1.20%</td>
<td>0.61%</td>
</tr>
<tr>
<td>CLASS 3</td>
<td>0.00%</td>
<td>2.23%</td>
<td>0.80%</td>
<td>0.62%</td>
<td>0.37%</td>
</tr>
<tr>
<td>CLASS 4</td>
<td>22.70%</td>
<td>25.31%</td>
<td>23.39%</td>
<td>25.08%</td>
<td>22.12%</td>
</tr>
<tr>
<td>CLASS 5</td>
<td>0.00%</td>
<td>0.51%</td>
<td>0.04%</td>
<td>0.06%</td>
<td>0.00%</td>
</tr>
<tr>
<td>CLASS 6</td>
<td>77.30%</td>
<td>67.47%</td>
<td>72.63%</td>
<td>72.24%</td>
<td>76.36%</td>
</tr>
</tbody>
</table>

The above example shows that our approach produces very close to the same results as the "true" optimal results under known cases.

6 Effect of Preferences

With the next example we will show what is the impact of having different preferences and we will compare the optimal results obtained by using our methodology, with results obtain from the standard mean variance approach.

Example

Suppose that we have two assets whose distributions have the same mean and variance but different higher moments. Let the monthly mean be 1% and the monthly standard deviation 4% (numbers consistent with the performance of S&P 500 over the last 10 years). The next table shows the descriptive statistics for the two assets. Figure (5) plots their densities. Also assume that the two assets are not highly correlated. (Asset 1 is S&P 500 and Asset 2 is generated to have the first 2 moments as Asset 1.)

<table>
<thead>
<tr>
<th></th>
<th>ASSET 1</th>
<th>ASSET 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN</td>
<td>1%</td>
<td>1%</td>
</tr>
<tr>
<td>ST. DEVIATION</td>
<td>4%</td>
<td>4%</td>
</tr>
<tr>
<td>SKEWNESS</td>
<td>-0.44</td>
<td>-7.71</td>
</tr>
<tr>
<td>KURTOSIS</td>
<td>1.15</td>
<td>74.74</td>
</tr>
<tr>
<td>MIN</td>
<td>-14%</td>
<td>-40%</td>
</tr>
<tr>
<td>MAX</td>
<td>11%</td>
<td>4%</td>
</tr>
<tr>
<td>CORRELATION</td>
<td>0.05</td>
<td></td>
</tr>
</tbody>
</table>
Since the correlation coefficient is so small, from a mean variance optimization objective point of view, the efficient frontier will be approximately 50% Asset 1 and 50% Asset 2. Suppose now that the investor’s objective is to maximize the probability of having positive returns, and in the same time he is very averse towards having returns below -5%. If he chooses to use the standard mean-variance approach, independent of his benchmark objectives, his optimal allocation will be always 50-50. Just by looking at the assets densities, it is clear that depending on the objective, one should hold either more of Asset 1 or Asset 2. If, for example, the objective is to have positive returns, then Asset 2 is the natural choice. On the other hand, if the objective is to be protected from big negative returns, the natural choice is Asset 1 (Asset 2 can produce large negative returns with a very small probability). The question is how to make the optimal decision. Our approach can identify, depending on the objective, the optimal combination of assets. Using the methodology described in the previous section, where $r_1 = 0\%$ and $r_2 = -5\%$, by varying the values for $\lambda$, we can produce a frontier. Figure 6 shows the frontier.

Note that the lowest point on the frontier corresponds to a portfolio that is very averse towards falling below -5%, hence the optimal allocation puts more weight on Asset 1, i.e. avoiding the possibility of a large negative return that could occur when holding Asset 2. The highest point on the frontier corresponds to a portfolio that is very aggressive in achieving positive returns, hence higher allocation in Asset 2. The mean-variance efficient portfolio 50-50 is also on the frontier. It is clear that the mean-variance allocation is a special case of our approach. We are enhancing it in two directions: asymmetric distributions and asymmetric utility functions.

7 Adding Hedge Funds to a Standard Portfolio

In this section, we look at adding hedge funds to a traditional portfolio. As discussed earlier, hedge fund returns are noted for their non-normal distributions. We use our methodology to capture the full value of the information contained in both tails of such a distribution. Consider
now that the investor is interested in including a hedge fund into his portfolio.

For US Equity and US Fixed Income we again use the long run statistics that generated the 60%-40% portfolio discussed earlier. For hedge funds, we use monthly aggregate index returns for the period 1995-2000. The index selected, the EACM 100, represents a broadly diversified portfolio of hedge funds. This particular index was selected because its returns are significantly less than those of the "database" indices. This index is based on a relatively stable set of established funds, In contrast, indices based on databases have higher returns because new funds are frequently added with months or years of return histories and funds are frequently dropped when they stop reporting to the database or close. The result is database indices tend to carry instant histories and survivor bias that is estimated at 1.5% to 3.0%. The historical characteristics for the 3 asset classes are:

<table>
<thead>
<tr>
<th></th>
<th>STOCKS</th>
<th>BONDS</th>
<th>HEDGE FUND</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>13.0%</td>
<td>6.0%</td>
<td>12.7%</td>
</tr>
<tr>
<td>Volatility</td>
<td>20.2%</td>
<td>8.7%</td>
<td>5.0%</td>
</tr>
<tr>
<td>Sharpe(at 5%)</td>
<td>0.4</td>
<td>0.1</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Before using our methodology, we looked at a simple mean-variance example with the three asset classes. As expected, it is not helpful with hedge funds. The high Sharpe ratio for the hedge funds causes them to dominate the allocation.

<table>
<thead>
<tr>
<th></th>
<th>STOCKS</th>
<th>BONDS</th>
<th>HEDGE FUND</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>1.4%</td>
<td>98.6%</td>
<td></td>
</tr>
</tbody>
</table>

While mathematically correct, a 99% allocation to hedge funds would not be considered rational. In fact, by only considering the mean and variance for each asset class, we are ignoring much valuable information contained in the series of monthly returns.

As we saw earlier, the normality assumption is reasonable for the US equity and fixed income returns but it is far from appropriate for the hedge fund returns. In addition, hedge fund index
data presents substantial problems with regard to survivor bias (Fung and Hsieh [3]) that inflates returns and autocorrelations that reduce observed volatility (Brooks and Kat [1]). Simply, hedge fund index data appears too good to be true, and is. Therefore, we systematically build a less optimistic return sample for hedge funds that incorporates the higher moments for the hedge fund asset class and uses more reasonable parameters for mean return and volatility. This new data set incorporates a healthy pessimism to challenge the highly optimistic allocations seen in mean-variance analysis.

For our distribution, we assume a mean return for hedge funds of 9.1%. This adjust for the full 3% survivor bias found in hedge fund index returns by Fung and Hsieh plus a significant downward revision to compensate for the strong bull market and associated capital market activity during the late 1990s. While hedge funds tend to have a low beta with the equity markets, they do tend to perform better when there is significant capital market activity in the form of merger activity, IPO issuance, restructuring, etc.

We also target a standard deviation of 9.0%. Brooks and Kat [1] suggest eliminating autocorrelation effects would increase volatility by up to 60%. Our own research using Bayesian tools to address estimation risk suggest an even higher standard deviation closer to 9.0%. Looking at historical data of various indices, we targeted skewness of -1 and kurtosis of 6. We use a mixture of two normal distributions to create a distribution that has the above four moments. Solving a nonlinear system of equations gives the values for the parameters of the mixture. The corresponding two normal distributions for the hedge fund and the resulting density plots for the three asset classes are:

*Insert Figure 7 Here*

*Insert Figure 8 Here*

The first figure shows the two normal distributions. Distribution 1 represents "steady state" returns when trading strategies are basically working as intended. It is characterized by a normal distribution with mean 14.9% and volatility of 6.2%, metrics similar to the HFR index data. Distribution 2 represents event risks or shocks to the steady state. It is a normal distribution
with mean -15.5% and volatility of 14.2%. The probability of choosing Distribution 1 is 81%.

The second figure shows the combined density for the simulated hedge fund data. It is created by assigning an 81% probability to returns coming from Distribution 1 and 19% from Distribution 2. This could be thought of as if one month in five, we see a strategy failure, e.g., deal breaks for merger arbitrage, equity market corrections for long-short strategies, or extraordinary interest rate volatility.

The third figure shows the densities for the three asset classes. We use normal distributions for the US Equity and Long-Term Bonds and the simulated distribution for the Hedge Fund.

*Insert Figure 9 Here*

With these three distributions for the three asset classes, we now look at incorporating hedge funds into a traditional portfolio. For our case study here, we look at an investor who not only seeks to exceed a 12% benchmark return, he also wants to avoid any losses for the portfolio overall. In our model, the investor’s true objective is to maximize the probability of outperforming a benchmark, \( r_1 = 12\% \) and his risk measure is the expected shortfall with respect to a benchmark, \( r_2 = 0\% \). In our formulation, the investor wants to solve the following problem:

\[
\max_{\bar{\omega} \in W} P(\bar{\omega}^T X > r) - \lambda E_X [r - \bar{\omega}^T X]^+ \\
r_1 = 12\% \\
r_2 = 0\%
\]

By varying the value of \( \lambda \), we can increase or decrease the investor’s aversion towards risk, which in this case is expressed in terms of having negative returns. If \( \lambda \) is made a very large number, the investor is only interested in avoiding losses (by analogy, the minimum variance portfolio in mean-variance optimizations). In this case, our methodology solves the optimal risk-averse portfolio to be:

**C: 10% Stocks, 70% Bonds and 20% Hedge Fund.**

By increasing \( \lambda \), we plot an efficient frontier with a vertical axis showing the probability of
outperforming \( r_1 \) and a horizontal axis of the risk term, in this case, the expected shortfall or the first partial moment. The risk term is scaled such that the largest value is equal to one.

*Insert Figure 10 Here*

The portfolio that is represented in red is 60% Stocks and 40% Bonds. It is clear that one can improve portfolio efficiency in two directions. First, the efficient portfolio that has the same probability of outperformance as the 60%-40% portfolio, but with less risk is portfolio:

**A: 30% Stocks, 28% Bonds, 42% Hedge Fund.**

Portfolio A preserves the probability of outperformance, but reduces the risk measure by 50%.

Second, the portfolio with the same risk but higher probability of outperformance is portfolio:

**B: 27% Stocks, 1% Bonds and 72% Hedge Fund.**

### 8 Who will NOT invest in Hedge Funds?

Next, let’s look at risk more broadly than simply aversion to losses, which is the first partial moment. Let’s consider a more general risk term that is the sum of the first \( K \) partial moments. In other words, assume that the investor is worried not only about the expected loss, but also about semivariance (placing a high penalty on any losses), conditional third moment (related to skewness), and conditional fourth moment (related to kurtosis - fear of extreme negative event).

\[
\max_{\omega \in W} P (\omega^T X > r) - \lambda E_X \left( [r - \omega^T X]^+ \right)^K
\]

The first case, that is for \( K = 1 \), was presented in the previous section. Using simulations, we can approximately identify the efficient frontiers for \( K \geq 2 \). The next three figures show the frontiers for \( K \) equal to 2, 3 and 4. From the figures you can see that as \( K \) increases the investor becomes more concerned about events far out in the left tail. As a consequence the allocation to hedge funds decreases. This is visually depicted by the position of the 60%-40% portfolio. As \( K \) increases, 60%-40% portfolio moves closer to the efficient frontier. One can speculate that
there will exist a value of $K$, probably very large, for which the 60%-40% portfolio will become efficient.

*Insert Figure 11 Here*

*Insert Figure 12 Here*

*Insert Figure 13 Here*

If we continue to expand the possible definitions of risk, it becomes clear that only an investor who seeks to avoid any extreme event with hedge funds will prefer the 60%-40% portfolio with no hedge funds to any portfolio with hedge funds. However, such an investor is willing to accept the variance inherent in the 60% allocation to stocks. Further, by making this choice he will reduce his chances of beating his benchmark of 12% per year.

Next tables show the allocation in the three asset classes for different risk measures. The portfolios in the first table are chosen from the set of efficient portfolios that have the same probability of outperforming 12% as the standard 60%-40% portfolio. We can see that by moving from the stocks to hedge funds reduces volatility substantially. As the risk measure changes ($K > 1$), the allocation to hedge funds decreases. However, in the case of $K = 4$, the allocation is 21%, still a substantial allocation. Thus, we conclude that as long as the investor is comfortable with being protected against the first four partial moments, an allocation to hedge funds is efficient. Of course, if the investor wants sure protection against rare events, the hedge fund asset class will not be an appropriate investment.

<table>
<thead>
<tr>
<th>$K$</th>
<th>Bonds</th>
<th>Stocks</th>
<th>Hedge Fund</th>
<th>% Reduction in Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = 1$</td>
<td>28%</td>
<td>30%</td>
<td>42%</td>
<td>58.1%</td>
</tr>
<tr>
<td>$K = 2$</td>
<td>31%</td>
<td>34%</td>
<td>35%</td>
<td>52.1%</td>
</tr>
<tr>
<td>$K = 3$</td>
<td>33%</td>
<td>39%</td>
<td>28%</td>
<td>46.1%</td>
</tr>
<tr>
<td>$K = 4$</td>
<td>38%</td>
<td>41%</td>
<td>21%</td>
<td>34.7%</td>
</tr>
</tbody>
</table>

In summary, our analysis shows that by transferring half of the allocation from Stocks to Hedge Fund, risk decreases relative to the risk of the 60%-40% portfolio.

The portfolios in the next table are chosen from the set of efficient portfolios that have the
same value as the risk measure for the standard 60%-40% portfolio. We see similar pattern in the allocation of hedge funds as \( K \) increases. The starting allocation in hedge fund \( K = 1 \) is much greater that the starting allocation in the previous table. This is due to the fact that now we are considering portfolios that will bring us closer to the goal of outperforming 12% per year. Hence, our risk aversion is low. Notice that this time the transformation of weight is from the fixed income class to the hedge fund. The last row of the table, \( K = 4 \), shows that the allocation in hedge fund is 38%. This is a point on the efficient frontier that corresponds to a smaller value of the risk aversion parameter compared to the value of the risk aversion parameter for the last portfolio in the previous table. At that point, the allocation to hedge funds is 21%. The increase in the probability of outperformance is mainly due to the increase in the allocation to hedge funds.

<table>
<thead>
<tr>
<th>( K )</th>
<th>Bonds</th>
<th>Stocks</th>
<th>Hedge Fund</th>
<th>% Increase in Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1%</td>
<td>27%</td>
<td>72%</td>
<td>9.0%</td>
</tr>
<tr>
<td>2</td>
<td>8%</td>
<td>35%</td>
<td>57%</td>
<td>6.0%</td>
</tr>
<tr>
<td>3</td>
<td>10%</td>
<td>43%</td>
<td>47%</td>
<td>5.0%</td>
</tr>
<tr>
<td>4</td>
<td>14%</td>
<td>48%</td>
<td>38%</td>
<td>4.0%</td>
</tr>
</tbody>
</table>

In summary, by transferring almost half of the allocation from fixed income to hedge funds, the probability of outperforming a 12% benchmark compared to the probability of the 60%-40% portfolio increases.

9 Conclusion

The return distributions of hedge funds and their indices are marked by their non-normal properties. This presents difficulties to traditional mean-variance models for asset allocation. Such models will tend to be overly optimistic in their allocations to hedge funds. In this paper, we introduce a more effective means of dealing with hedge fund portfolios. Using procedures from stochastic programming, this model is able to recognize and use the information embedded in the unusual return distributions of hedge funds.
We applied the model to hedge funds to examine the effects of semi-variance, conditional third and fourth moments on portfolio allocations to hedge funds. The results indicate a substantial allocation to hedge funds is justified even with consideration for the highly unusual kurtosis and skewness.

References


Figure 1 shows the histogram for S&P 500 monthly returns for the period 1996-2001.

Figure 2 shows the histogram for the Lehman Aggregate Bond Index monthly returns for the period 1996-2001.
Figure 3 shows the histogram of Merger arbitrage hedge fund monthly returns with assets over $1 billion and with history of greater than five years.

Figure 4 shows the histogram of Long/Short hedge fund monthly returns with assets over $1 billion and with history of greater than five years.
Figure 5 plots the densities for the two assets for the Example. Asset 1 has a monthly mean of 1% and a monthly standard deviation of 4%; Asset 2 has the same mean and standard deviation, additionally Asset 2 has a negative skewness and high kurtosis.

Figure 6 shows the efficient frontier produced for the Example. The x-axis is the Expected Shortfall with respect to (-5%), the y-axis is the Probability of outperforming 0%.
Distribution 1 represents "steady state" returns when trading strategies are basically working as intended. It is characterized by a normal distribution with mean 14.9% and volatility of 6.2%, metrics similar to the HFR index data. Distribution 2 represents event risks or shocks to the steady state. It is a normal distribution with mean -15.5% and volatility of 14.2%. The probability of choosing Distribution 1 is 81%.

Figure 8 shows the combined density for the simulated hedge fund data. It is created by assigning an 81% probability to returns coming from Distribution 1 and 19% from Distribution 2.
Figure 9 shows the densities for the three asset classes. We use normal distributions for the US Equity and Long-Term Bonds and the mixture of two Normal distributions for the Hedge Fund.

Figure 10 plots an efficient frontier with a vertical axis showing the probability of outperforming $r_1$ and a horizontal axis of the risk term, in this case, the expected shortfall or the first partial moment. The risk term is scaled such that the largest value is equal to one.
Figure 11 plots an efficient frontier with a vertical axis showing the probability of outperforming $r_1$ and a horizontal axis of the risk term, in this case, the second partial moment. The risk term is scaled such that the largest value is equal to one.

Figure 12 plots an efficient frontier with a vertical axis showing the probability of outperforming $r_1$ and a horizontal axis of the risk term, in this case, the third partial moment. The risk term is scaled such that the largest value is equal to one.
Figure 13 plots an efficient frontier with a vertical axis showing the probability of outperforming $r_1$ and a horizontal axis of the risk term, in this case, the fourth partial moment. The risk term is scaled such that the largest value is equal to one.