Ground vehicle system dynamics – 1

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Overview

A study of ground vehicle systems builds on fundamental laws and principles from dynamics. Our goal in this course is to develop mathematical (systems) models that help us understand, analyze, and control critical modes of vehicle motion.

We want to derive vehicle models so we can study the dynamic response under prescribed conditions and inputs, usually solved using computer simulation.

For this reason, we distinguish between models derived to solve for specific variables at a given time or orientation of a vehicle, and instead derive equations that allow us to solve for key variables over all time.
It will be assumed that you have completed an introductory course in dynamics, or have sufficient physics background to understand the following discussions. This overview focuses on topics that are particularly useful when studying ground vehicle systems.

A two-axle vehicle in acceleration

FBD:

A two-axle vehicle on an incline

FBD:
Concepts useful in vehicle dynamics

- Coordinate systems used for vehicle motion modeling
- How to express position vectors in defined coordinate systems, and how to differentiate to derive velocity and acceleration expressions
- Particle and rigid body kinematics
- Relative velocity/acceleration, transferring between coordinate systems
- Free body (or force) diagrams; static analysis
- Mass properties (e.g., moments of inertia, inertia matrix, etc.)
- Newton’s laws, Euler’s equations
- Models for induced forces and torques, due to friction or elastic effects, as well as applied forces and torques due to external sources

- Coordinate transformations are essential for some problems (e.g., turning) – to be reviewed in next lecture
- Bond graphs (a topic covered in ME 344 and ME 383Q-modeling) can be helpful mostly in powertrain problems, but are not necessary for this course
Vehicle-fixed coordinate system; SAE standard

Ground vehicle coordinate systems commonly employ a coordinate system standardized by SAE*.

Consider the standard SAE coordinate system and terminology.

SAE vehicle axis system
- $x =$ forward, on the longitudinal plane of symmetry
- $y =$ lateral out the right side of the vehicle
- $z =$ downward with respect to the vehicle
- $p =$ roll velocity about the $x$ axis
- $q =$ pitch velocity about the $y$ axis
- $r =$ yaw velocity about the $z$ axis

*SAE = Society of Automotive Engineers

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Earth-fixed (or global) coordinate system

- **X** = forward travel
- **Y** = travel to the right
- **Z** = vertical travel (+down)
- **ψ** = heading angle (between x and X in ground plane)
- **ν** = course angle (between vehicle velocity vector and X axis)
- **β** = sideslip angle (between x and vehicle velocity vector)
Relative velocity of particles

Review these kinematic concepts as needed in a reference of your choice. For example: Meriam & Kraige, Karnopp & Margolis (see references list).
Relative acceleration of particles

You can show by successive differentiation

\[ \vec{a}_A = \vec{a}_B + \vec{\omega} \times \vec{r} + \vec{\omega} \times \vec{v} + \vec{v}_{rel} \]

\[ \vec{v}_{rel} = \vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel} \]

\[ \vec{a}_A = \vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \left( 2\vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel} \right) \]

\[ \text{normal} \quad \text{translational} \quad \text{Coriolis} \quad \text{rel. to moving coord. sys. that is rotating \at \omega, \omega} \]

Review these kinematic concepts as needed in a reference of your choice. For example: Meriam & Kraige, Karnopp & Margolis (see references list).
Translating and rotating ref. frames

It is helpful to have an understanding of the coordinate systems used for rigid body analysis, and the terminology employed for these applications. Key result: transformation of a time derivative of a vector quantity can be any vector quantity.

\[
\frac{d\mathbf{V}}{dt}_{\text{XYZ}} = \left( \frac{d\mathbf{V}}{dt} \right)_{\text{xyz}} + \dot{\Omega} \times \mathbf{V}
\]

This relationship between vector quantities in xyz and XYZ will prove very useful.

\(\mathbf{V}\) can be any vector quantity.

Ref. Meriam & Kraige, Ch. 7 or Karnopp & Margolis, Section 1.4
**Example: Karnopp and Margolis P1.11**

Figure P1.11 shows the top view of a vehicle that has mass $m$ and c.g. moment of inertia about the axis out of the page, $I_g$. The center of gravity is located a distance $a$ from the front axle and a distance $b$ from the rear axle. The half-width of the vehicle is $w/2$. The front wheels can be steered, indicated by the steer angle $\delta$. A body-fixed coordinate frame is attached to the vehicle at its center of gravity and aligned as shown. The body-fixed velocity components of the center of gravity and the yaw angular velocity are indicated.

(a) Using arrows and symbols, transfer the c.g. velocity to body-fixed directions at the four wheels.

(b) If each wheel is constrained to have no velocity perpendicular to the plane of the wheel, state the kinematic constraints for each wheel.

**NOTE**: type of kinematics knowledge that can be helpful in vehicle applications

\[
\ddot{V}_p = \dot{V}_o + \ddot{\Omega} \times \dot{R} \\
\ddot{\vec{A}}_p = \ddot{\vec{A}}_o + \dot{\ddot{\Omega}} \times \dot{R} + \dot{\ddot{\Omega}} \times (\dddot{\Omega} \times R)
\]

We will return to this example in the next lecture.

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Newton’s 2\textsuperscript{nd} Law: You are familiar with this fundamental principle of dynamics, typically stated as ‘\textit{f-equals-m-a}’.

It is understood that this equation can be used in different ways.

My suggestion is that you always think of this law as one that expresses the \textit{change in momentum of a body, p, due to imposed forces}.

Recall definition of (translational) momentum:

So Newton’s law is:

\[
p = mv
\]

\[
\frac{dp}{dt} = m \frac{dv}{dt} = ma = F
\]

\textit{cause} this \textbf{net forces on a body}

This defines a \textbf{dynamic equation}. A model. The \textbf{hard part} is figuring out the \textit{F}.
AXIOMS, OR
LAWS OF MOTION¹

LAW I
Every body continues in its state of rest, or of uniform motion in a right
line, unless it is compelled to change that state by forces impressed upon it.

Projectiles continue in their motions, so far as they are not retarded
by the resistance of the air, or impelled downwards by the force of
gravity. A top, whose parts by their cohesion are continually drawn
aside from rectilinear motions, does not cease its rotation, otherwise than
as it is retarded by the air. The greater bodies of the planets and comets,
meeting with less resistance in freer spaces, preserve their motions both
progressive and circular for a much longer time.

LAW II²
The change of motion is proportional to the motive force impressed; and
is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the
motion, a triple force triple the motion, whether that force be impressed
altogether and at once, or gradually and successively. And this motion
(being always directed the same way with the generating force), if the
body moved before, is added to or subtracted from the former motion,
according as they directly conspire with or are directly contrary to each
other; or obliquely joined, when they are oblique, so as to produce a new
motion compounded from the determination of both.

LAW III
To every action there is always opposed an equal reaction: or, the mutual
actions of two bodies upon each other are always equal, and directed to
contrary parts.

Whatever draws or presses another is as much drawn or pressed by that
other. If you press a stone with your finger, the finger is also pressed by the

¹ Appendix, Note 14. ² Appendix, Note 15.

\[ \frac{dp}{dt} = m \frac{dv}{dt} = \sum F \]

Causal relation:
Forces induce accelerations, which we integrate to get velocities and
positions. It is the latter two that we need in order to define the forces
themselves!
What’s on the right hand side?

- External forces from environment, road, etc.
- Forces from vehicle subsystem components such as from suspension force elements (shocks, springs, active suspension actuators, etc.)
- Forces induced between *running gear* and terrain, transmitted into the chassis (body)
- Let’s list some of these.
Class discussion: common force elements

- Force element, where used, depends on (velocity, position, etc.); sketch the *constitutive behavior*
- etc.

<To be discussed and filled in during class>
Full 3D for a rigid body:

\[ F = \left. \frac{dp}{dt} \right|_{xyz} + \Omega \times p \]

\[ T = \left. \frac{dh}{dt} \right|_{xyz} + \Omega \times h \]

\[ F_x = \dot{p}_x + \Omega_y p_z - \Omega_z p_y \]
\[ F_y = \dot{p}_y + \Omega_z p_x - \Omega_x p_z \]
\[ F_z = \dot{p}_z + \Omega_x p_y - \Omega_y p_x \]

\[ T_x = \dot{h}_x + \Omega_y h_z - \Omega_z h_y \]
\[ T_y = \dot{h}_y + \Omega_z h_x - \Omega_x h_z \]
\[ T_z = \dot{h}_z + \Omega_x h_y - \Omega_y h_x \]

These are sometimes referred to as the Euler equations, often only when you let the reference axes coincide with the principal axes of inertia at the mass center or at a point fixed to the body so the products of inertia go to zero – this leads to a simpler form.
By using **body fixed coordinates**, the rotational inertial properties remain fixed.

The products of inertia* are all zero, and this makes it convenient for our purposes.

*See dynamics text to review inertial properties.

Define:

$$p = m v$$  \hspace{1cm} \text{(translational momentum)}$$

Then:

$$\frac{dp}{dt}_{XYZ} = \frac{dp}{dt}_{xyz} + \Omega \times p$$

With \(v\) relative to rotating frame.

Newton’s law:

$$\left( \frac{dp}{dt} \right)_{XYZ} = F$$

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• **State space** formulation for dynamic states
• Fundamentally, these are in **momentum** states
• Usually switch to **velocity** states

\[
\begin{align*}
\dot{p}_x &= m\dot{v}_x = F_x - \Omega_y p_z + \Omega_z p_y \\
\dot{p}_y &= m\dot{v}_y = F_y - \Omega_z p_x + \Omega_x p_z \\
\dot{p}_z &= m\dot{v}_z = F_z - \Omega_x p_y + \Omega_y p_x \\
\dot{h}_x &= I_x \dot{\omega}_x = T_x - \Omega_y h_z + \Omega_z h_y \\
\dot{h}_y &= I_y \dot{\omega}_y = T_y - \Omega_z h_x + \Omega_x h_z \\
\dot{h}_z &= I_z \dot{\omega}_z = T_z - \Omega_x h_y + \Omega_y h_x
\end{align*}
\]
By having the full dynamic equations at your disposal, you can:

- Make sure to include effects that might be hard to ‘see’ intuitively or reliably

- You can ‘throw out’ (prune) terms that do not apply, based on your assumptions, and keep those that will impact the problem at hand.
Static and equilibrium problems

Equilibrium problems refer to those where there are no dynamic effects (static, or system has reached dynamic equilibrium). In the case of vehicle dynamics, we can represent this mathematically by making all the rates of change of the 6 DOF (the left hand side) equal to zero; i.e.,

\[
\begin{align*}
\dot{p}_x &= 0 \\
\dot{p}_y &= 0 \\
\dot{p}_z &= 0 \\
\dot{h}_x &= 0 \\
\dot{h}_y &= 0 \\
\dot{h}_z &= 0
\end{align*}
\]

This would allow us to write up to six equations (based on the right hand side – i.e., the force relations) to solve for any unknown variables. Usually, we need only consider three mutually perpendicular axes, like the body fixed axes, but in some cases it may be more convenient to work with other coordinate axes.

Also, if the problem is simple enough, the approach does not have to be so general.

Effectively, this is what you do when you solve ‘static’ problems.
**Example: 2 axle vehicle, 1 DOF**

A two-axle vehicle with mass, $m_v$, at center $G$ is in maximum acceleration state. Find expressions for the total normal forces at the front and rear pairs of wheels, $N_f$ and $N_r$, respectively. Assume the mass of the wheels is small compared with the total mass of the car, and that the coefficient of static friction between the road and the rear driving wheels is $\mu$.

$$a_x = x\text{-component of acceleration}$$

$$m = \text{mass}$$

$$N_r = \text{weight on rear wheels}$$

$$N_f = \text{weight on front wheels}$$

$$\mu = \text{coefficient of friction}$$

$$h = \text{height of c.g.}$$
Sliding and rolling friction

- Friction in sliding and/or rolling plays a critical role in ground vehicle applications.

- Ogata handout – a good review before we begin looking at the tire-terrain interface, which can be very complex.
  - Simple translation: two surfaces sliding, concept of static and sliding friction
  - Friction force as a function of relative velocity

- Rolling friction vs. rolling resistance. Resistance to motion that takes place when an object rolls over an abutting surface
  - Slip at contact region?
  - Other losses (e.g. in material)?

This is a good illustration of the mechanics that lead to ‘rolling resistance’.
6/5 The 1650-kg car has its mass center at \( G \). Calculate the normal forces \( N_A \) and \( N_B \) between the road and the front and rear pairs of wheels under conditions of maximum acceleration. The mass of the wheels is small compared with the total mass of the car. The coefficient of static friction between the road and the rear driving wheels is 0.8.

Ans. \( N_A = 6.85 \text{ kN} \), \( N_B = 9.34 \text{ kN} \)

Solution from MK

\[ mg = 1650 \times (9.81) = 16.19 \times (10^3) \text{ N} \]

\[ F + \sum M_c = \text{mad} = 0 : N_B (2.4) - 0.8 N_B (0.4) - 16.19 (10^3) 1.2 = 0 \]

\[ N_B = 9.34 (10^3) \text{ N} \text{ or } N_B = 9.34 \text{ kN} \]

\[ \sum F_y = 0 : N_A + 9.34 (10^3) - 16.19 (10^3) = 0 \]

\[ N_A = 6.85 (10^3) \text{ N} \text{ or } N_A = 6.85 \text{ kN} \]
Example: finding reaction forces for two-axle vehicle in maximum acceleration

A two-axle vehicle with mass, \( m_v \), at center \( G \), is in maximum acceleration state. Find expressions for the total normal forces at the front and rear pairs of wheels, \( N_f \) and \( N_r \), respectively. Assume the mass of the wheels is small compared with the total mass of the car, and that the coefficient of static friction between the road and the rear driving wheels is \( \mu \).

Numerical calculations:

\[
\begin{align*}
  m_v &:= 1650 \cdot \text{kg} \\
  h &:= 0.4 \cdot \text{m} \\
  l_1 &:= 1.2 \cdot \text{m} \\
  l_2 &:= 1.2 \cdot \text{m} \\
  \mu &:= 0.8 \\
  W_v &:= m_v \cdot g = 16181 \text{N} \\
  L_t &:= l_1 + l_2 = 2.4 \text{m} \\
  N_r &:= \frac{l_1}{L_t - \mu \cdot h} \cdot W_v = 9335.2 \text{N} \\
  N_f &:= \frac{l_2 - \mu \cdot h}{L_t - \mu \cdot h} \cdot W_v = 6845.8 \text{N}
\end{align*}
\]
Example: reducing the 6 DOF equations for a 1 DOF vehicle
Reduce the Euler equations to the form needed in this problem.
Note how you can define the static problems to find unknowns.

\[
\begin{align*}
\dot{p}_x &= F_x - \Omega_y p_z + \Omega_z p_y \\
\dot{p}_y &= F_y - \Omega_z p_x + \Omega_x p_z \\
\dot{p}_z &= F_z - \Omega_x p_y + \Omega_y p_x \\
\sum F_z &= 0 = -N_f - N_r + W
\end{align*}
\]

\[
\begin{bmatrix}
1 & 1 \\
-l_1 & l_2 - \mu h
\end{bmatrix}
\begin{bmatrix}
N_f \\
N_r
\end{bmatrix}
= 
\begin{bmatrix}
W \\
0
\end{bmatrix}
\Rightarrow
\begin{align*}
N_f &= \frac{l_2 - \mu h}{L - \mu h} W \\
N_r &= \frac{l_1}{L - \mu h} W \\
W &= m_v g
\end{align*}
\]

These two equations give two equations, two unknowns. Solve for the unknown forces, then apply to x-direction translational equation to find rate of change of forward velocity (acceleration).
Example: solving for acceleration of powered mower; given friction

6/2 The rear-wheel-drive lawn mower, when placed into gear while at rest, is observed to momentarily spin its rear tires as it accelerates. If the coefficients of friction between the rear tires and the ground are $\mu_s = 0.7$ and $\mu_k = 0.5$, determine the forward acceleration $a$ of the mower. The mass of the mower and attached bag is 50 kg with center of mass at $G$. Assume that the operator does not push on the handle, so that $P = 0$.

Solution from Meriam and Kraige

NOTE: at the end, you should feel comfortable with these types of problems

However, you should be able to work the problem with a system dynamics approach.
**Alternative approach** by application of Newton-Euler (N-E) equations:

Running the numbers leads to same results...

\[ h = 215\text{-mm} \quad m_v = 50\text{-kg} \]
\[ L_1 = 500\text{-mm} \quad L_y = 0 \quad (\text{not given or needed}) \]
\[ L_2 = 200\text{-mm} \quad \mu = 0.5 \]
\[ L_t := L_1 + L_2 \]

This model says the vehicle will operate with constant acceleration.

Is this realistic?
We can model **rolling resistance** (RR) forces as applied at the wheel centers or at the vehicle CG. If RR forces are placed at the wheel, we need more information about the vehicle geometry (e.g., wheel diameter) to build a dynamic model.

We’ll discuss these issues further and study how the equations are influenced when we begin building full performance models for road vehicles.
Example: dynamic response analysis of the powered-mower

The powered-mower example is useful for illustrating vehicle performance evaluation. First, we add loading effects on the vehicle applied on the wheels and/or body. In the translational momentum equation (x-direction),

\[
\dot{p}_x = m \frac{dv_x}{dt} = \sum F_x = F_t - \sum \text{loads}
\]

Assume a net load force that is linearly proportional to velocity and applied at the CG: \( F_r = B \cdot v_x \)

\[ \therefore m \dot{v}_x = \sum F_x = F_t - F_f = F_t - Bv_x \]

Now vehicle speed can be modeled by a 1\textsuperscript{st} order linear ODE, in standard form,

\[ m \dot{v}_x + B v_x = F_t \implies \tau \dot{v}_x + v_x = u_d \]

At this point, standard solutions give the transient response. For example, for \( F_t \) constant, the velocity increases with typical exponential response to a steady-state (final) value.

\[ v_x(t) = u_d \left[ 1 - \exp \left( -t/\tau \right) \right] \]

assuming: \( v_x(t = 0) = 0 \)
Example: Matlab simulation of powered-mower performance

function [xdot, y] = powered_mower_1(t, x);
global h L1 L2 L mv mu g Iy B % physical state variables
vx = x(1);

% calculate rear and front
% contact/reaction forces
% note: constant if Fr is at CG
Nr = L1*mv*g/(L-mu*h);
Nf = (L2-mu*h)*mv*g/(L-mu*h);
% forces on vehicle
Fb = B*vx;
Ft = mu*Nr;

% state equations
vx_dot = (Ft-Fb)/mv;
xdot = [vx_dot];

% output variables
y(1) = Nr; % rear axle load
y(2) = Nf; % front axle load
y(3) = vx_dot/g; % accel in g units
y(4) = Ft; % rear traction force
Steady-state analysis – graphical

If the constant $B$ is known, it is possible to find a steady-state speed. For this simple case, the steady state solution is found easily from the ODE by setting the rate of change of velocity to zero (defining steady-state or equilibrium condition). Alternatively, the steady-state solution can be found graphically. Both are illustrated below.

In original case (no RR):

$$F_t = \frac{\mu l}{L - \mu h} W = \text{constant}$$

If $F_t$ was not constant, but depended on the velocity, this method still applies as shown.

Plot both $F_t$ and the load $F_r$ on the same force-velocity graph, and find the intersection. This is a solution to a source-load analysis.

Note: $v_{ss} = u_d$
Summary

- This was a first look into how dynamics principles are used in study of ground vehicle dynamics.
- We reviewed concepts from kinematics (motion) and dynamics as applied to ground vehicle problems.
- We seek to formulate models in a system dynamics (state space) form so we can apply common computer simulation tools.
- Steady-state analysis can also be easily applied, and sometimes the results are very useful in analyzing practical problems.
- In the next lecture, we discuss additional concepts useful for ground vehicle systems study: rolling/slipping of wheels and coordinate transformations.
- See the attached problems with solutions
Suggested References and Reading

• Hibbeler, Engineering Mechanics: Dynamics, 9th ed., Prentice-Hall. Introductory text often used in course like ME 324. Look for any examples and problems that have wheels.


• Karnopp, D.C., and D.L. Margolis, Engineering Applications of Dynamics, John Wiley & Sons, New York, 2008. An excellent intermediate engineering dynamics text focused on building mathematical models. Good overview of fundamental material and provides a deeper understanding beyond a traditional sophomore level course in ME dynamics (the road to systems).

• Wong, J.Y., Theory of Ground Vehicles, John Wiley and Sons, Inc., New York, 1993 (and later editions up to 2010). One of the current standard books for engineers working in all areas of ground vehicle systems. Relevant but assumes reader has background in fundamental dynamics principles.
Problems with solutions

The following problems illustrate some of the concepts discussed in these slides.

• Example 1: Finding height of center of gravity of a two-axle (rigid) vehicle
• Example 2: Weight distribution in a three-wheeled vehicle on level ground
• Example 3: Problem 6/3 (Meriam & Kraige); application of D’Alembert Approach
• Exercise Problem 1: exercise problem for a two-axle vehicle in acceleration up a grade
Example 1: Finding height of center of gravity of a two-axle (rigid) vehicle

You can find the height of the center of gravity above the axle length as follows. Assume a rigid suspension.

First:
\[ R_4 (c + d) = Wc \]

\[ c = AB - CD \]

\[ = a \cos \theta - (H - r) \sin \theta \]

Then,

\[ R_4 L \cos \theta = Wa \cos \theta - W (H - r) \sin \theta \]

\[ \Rightarrow (H - r) = \left( \frac{Wa - R_4 L}{W} \right) \cot \theta \]

where: \( \sin \theta = \frac{h}{L} \)
Example 2: Weight distribution in a three-wheeled vehicle on level ground

Taking moments about the R-R axis you can show that,

\[ +\text{ccw} \sum T_R = 0 = F_F L - W l_2 \Rightarrow F_F = \frac{l_2}{L} W \]

Then about C-C (looking from the front),

\[ +\text{ccw} \sum T_C = 0 = + W e + F_{LR} d - F_{RR} d \]

\[ \Rightarrow (F_{RR} - F_{LR}) = \frac{e}{d} W \]

And then about F-F,

\[ +\text{ccw} \sum T_F = 0 = + W l_1 - (F_{RR} + F_{LR}) L \]

\[ \Rightarrow (F_{RR} + F_{LR}) = \frac{l_1}{L} W \]

Solving then for the rear axle forces,

\[ F_{RR} = \left( \frac{e}{d} + \frac{l_1}{L} \right) \frac{W}{2} \]

\[ F_{LR} = \left( \frac{l_1}{L} - \frac{e}{d} \right) \frac{W}{2} \]
Example 3: another example using moment about arbitrary point

Recall Meriam & Kraige, Problem 6/3
A bicyclist applies the brakes as he descends a $10^\circ$ incline.

What deceleration $a$ would cause the dangerous condition of tipping about the front wheel $A$?

The combined center of mass of the rider and bicycle is at $G$. Ans. $a = 0.510g$

Tipping at front wheel: $N_B, F_B \Rightarrow 0$

$$\Sigma M_A = ma : \ mg \ (25 \cos 10^\circ - 36 \sin 10^\circ) = ma \ (36)$$

Solve to obtain $a = 0.510g \ (16.43 \text{ ft/sec}^2)$
Example 3: alternative approach to the tipping cyclist problem

A cyclist applies brakes descending down an incline. At what level of deceleration would tipping about front wheel occur.

\[ F_{f} = \frac{l_{f}}{h} \cdot N_{f} = \frac{l_{f}}{h} \cdot W \cos \theta \]

\[ F_{x} = m \ddot{u}_{x} + F_{f} = W \sin \theta \]

\[ + h F_{f} - l_{f} N_{f} = 0 \]

\[ \ddot{u}_{x} = \frac{1}{m} \left[ W \sin \theta - \frac{l_{f}}{h} W \cos \theta \right] \]

\[ = g \left[ \sin \theta - \frac{l_{f}}{h} \cos \theta \right] \]

If you were given a value for (static) coefficient of friction, can you find a condition/requirement that would need to be satisfied in order for tipping to occur?
Problem 1: Consider a car with wheelbase $L = 2.44 \text{ m}$, front axle weight $N_f = 3986 \text{ N}$, rear axle weight $N_r = 5978 \text{ N}$, and center of gravity $h = 1.22 \text{ m}$ above ground level.

(a) If this car accelerates at $1.52 \text{ m/s}^2$ up a gradient of 1 in 10, what will be the wheel-road reactions normal to the road at the front and back? (ans. $2696 \text{ N}$, $7219 \text{ N}$)

(b) If the car is driven by the rear wheels only, what must be the minimum value of coefficient of friction to enable the acceleration to be obtained without wheel slip? (ans. $0.35$)

(c) What would be the value of the acceleration that would cause the car to be about to overturn? (ans. $6.83 \text{ m/s}^2$)